

LOCAL FRACTIONAL HELMHOLTZ SIMULATION FOR HEAT CONDUCTION IN FRACTAL MEDIA

by

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In this paper, we consider the generalized local fractional 2-D Helmholtz equation in steady heat transfer process, which can be used to model the steady-state heat conduction in fractal media. The Yang-Fourier transform and Yang-Laplace transform method are used to solve the equation. The integral expression of the solutions is obtained in detail.

Key words: local fractional Helmholtz equation, Yang-Fourier transform, Yang-Laplace transform, steady heat transfer

Introduction

In the present investigation, we study the following local fractional 2-D Helmholtz equation in the steady heat transfer process, defined:

$$\frac{\partial^{2\alpha} u(x, y)}{\partial x^{2\alpha}} + \frac{\partial^{2\beta} u(x, y)}{\partial y^{2\beta}} + k^2 u(x, y) = f(x, y) \quad (1)$$

with the conditions:

$$u(x, 0) = \varphi(x), \frac{\partial u(x, 0)}{\partial y} = \psi(x), \lim_{x \rightarrow \pm\infty} u(x, y) = 0, \quad (2)$$

where k is a constant, $\partial^{2\alpha} u / \partial x^{2\alpha}$ and $\partial^{2\beta} u / \partial y^{2\beta}$ are the local fractional derivatives [1] ($0 < \alpha \leq 1$, $0 < \beta \leq 1$), $\varphi(x)$, $\psi(x)$, and $f(x, y)$ are given functions.

The classical Helmholtz equation arise naturally in many physical applications such as elastic waves in solids including vibrating string, bars, membranes, electromagnetic waves, and the heat conduction in nuclear reactors [2, 3]. However, the classical calculus cannot be used to deal with some non-differentiable problems. The local fractional calculus is a powerful tool for studying them [4-17]. Here we use the local fractional 2-D Helmholtz eq. (1) to model the steady-state heat conduction in fractal media [18, 19].

The Helmholtz equation involving local fractional derivative operators have been investigated over the last decade [20, 21]. In case of $\alpha = \beta$, the eq. (1) have been solved by applying the local fractional series expansion method and the local fractional variational iteration method, but they obtained only the approximate analytic solutions [22].

The main objective of the present paper is to solve the problems eqs. (1) and (2) by means of the Yang-Fourier transform and Yang-Laplace transform method [23, 24].

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Preliminaries

In this section, we give some definitions and properties of local fractional derivative, Yang-Fourier transform and Yang-Laplace transform, for more detail see [1, 23, 24].

Definition 1. For arbitrary $\varepsilon > 0$, the relation:

$$|f(x) - f(x_0)| < \varepsilon^\alpha \quad (3)$$

exists with $|x - x_0| < \delta$. Then, $f(x)$ is called local fractional continuous at x_0 , which is denoted by $\lim f(x) = f(x_0)$. If $f(x)$ is local fractional continuous on the interval (a, b) we denote:

$$f(x) \in C_\alpha(a, b)$$

Definition 2. Let $f(x) \in C_\alpha(a, b)$ In fractal space, the local fractional derivative of $f(x)$ of order α at the point $x - x_0$ is given [1]:

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x) \Big|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \quad (4)$$

where

$$\Delta [f(x) - f(x_0)] \cong \Gamma(\alpha + 1) [f(x) - f(x_0)]$$

The local fractional partial derivative of high order is defined in the form [1]:

$$\frac{\partial^{k\alpha} f(x, t)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x, t) \quad (5)$$

Definition 3. A partition of the interval $[a, b]$ is denoted as $(t_j, t_{j+1}), j = 0, 1, \dots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max \{\Delta t_0, \Delta t_1, \dots, \Delta t_{N-1}\}$. The local fractional integral of $f(x)$ in the interval $[a, b]$ is defined [1]:

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(x) (dx)^\alpha = \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha \quad (6)$$

We list two useful formulas of local fractional derivatives and integral [1]:

$$\frac{d^\alpha (x^{n\alpha})}{dx^\alpha} = \frac{\Gamma(1 + n\alpha) x^{(n-1)\alpha}}{\Gamma(1 + (n-1)\alpha)} \quad (7)$$

$$\frac{1}{\Gamma(1 + \alpha)} \int_a^b x^{n\alpha} (dx)^\alpha = \frac{\Gamma(1 + n\alpha) [b^{(n+1)\alpha} - a^{(n+1)\alpha}]}{\Gamma[1 + (n+1)\alpha]} \quad (8)$$

Definition 4. In the fractal space, the Mittag-Leffler function is defined:

$$E_\lambda(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{(n\alpha)}}{\Gamma(1 + n\lambda)}, \quad 0 < \alpha \leq 1 \quad (9)$$

Definition 5. Let:

$$\frac{1}{\Gamma(1 + \alpha)} \int_0^\infty |f(x)| (dx)^\alpha < k < \infty, \quad 0 < \alpha \leq 1$$

The Yang-Laplace transforms of $f(x)$ is defined:

$$L_\alpha \{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha \quad (10)$$

where the latter integral converges and $s^\alpha \in R^\alpha$.

Definition 6. The inverse transform of the Yang-Laplace transforms of $f(x)$ is defined:

$$L_\alpha^{-1} \{f_s^{L,\alpha}(s)\} = f(x) = \frac{1}{(2\pi)^\alpha} \int_{\beta-i\infty}^{\beta+i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(S) (ds)^\alpha \quad (11)$$

where $s^\alpha = \beta^\alpha + i^\alpha \bar{\omega}^\alpha$, fractal imaginary unit i^α and $\text{Re}(s) = \beta > 0$.

The basic properties of local fractional Yang-Laplace transform are given.

We have the following formulas:

$$L_\alpha \{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bf_s^{L,\alpha}(s), \quad L_\alpha \{E_\alpha(c^\alpha x^\alpha)\} = \frac{1}{s^\alpha - c^\alpha}, \quad L_\alpha \{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}}$$

$$L_\alpha \{E_\alpha(c^\alpha x^\alpha) f(x)\} = f_s^{L,\alpha}(s-c), \quad L_\alpha \{f^{(\alpha)}(x)\} = s^\alpha f_s^{L,\alpha}(s) - f(0)$$

$$L_\alpha^{-1} \left[\frac{s^{\zeta-\nu(2-\mu)}}{s^\mu + r} \right] (x) = x^{1-(1-\nu)(2-\mu)-\zeta} E_{\mu,2-(1-\nu)(2-\mu)-\zeta}(-rx^\mu)$$

where $1 < \mu \leq 2$, $0 \leq \nu \leq 1$, $\zeta > 0$.

Definition 7. If $f(x) \in C_\alpha(a, b)$ then the Yang-Fourier transform is defined:

$$F_\alpha[f(x); \omega] = f_\omega^{F,\alpha}(\omega) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{+\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha) f(x) (dx)^\alpha \quad (12)$$

and its inverse Yang-Fourier transform:

$$f(x) = F_\alpha^{-1}[f_\omega^{F,\alpha}(\omega)] = \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) f_\omega^{F,\alpha}(\omega) (d\omega)^\alpha \quad (13)$$

The following formulas:

$$F_\alpha[af(x) + bg(x); \omega] = aF_\alpha[f(x); \omega] + bF_\alpha[g(x); \omega] \quad (14)$$

$$F_\alpha \left[\frac{d^\alpha}{dx^\alpha} h(x); \omega \right] = i^\alpha \omega^\alpha F[h(x); \omega] \quad (15)$$

hold true, where $\lim_{x \rightarrow \pm\infty} h(x) = 0$.

The solution of the problem (1)-(2)

In this section, we consider the following problem of local fractional 2-D Helmholtz equation:

$$\begin{cases} \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \frac{\partial^{2\beta} u}{\partial y^{2\beta}} + k^2 u(x, y) = f(x, y) \\ \frac{\partial u(x, 0)}{\partial y} = \psi(x), \quad u(x, 0) = \varphi(x), \quad \lim_{x \rightarrow \pm\infty} u(x, y) = 0 \end{cases} \quad (16)$$

where the functions $f(x, y)$, $\varphi(x)$, and $\psi(x)$ are local fractional continuous.

Let:

$$L_\beta[u(x, y); s^\beta] = \bar{u}(x, s^\beta), \quad L_\beta[f(x, y); s^\beta] = \bar{f}(x, s^\beta)$$

It follows that:

$$\frac{\partial^{2\alpha} \bar{u}(x, s^\beta)}{\partial x^{2\alpha}} + s^{2\beta} \bar{u}(x, s^\beta) - s^\beta \varphi(x) - \psi(x) + k^2 \bar{u}(x, s^\beta) = \bar{f}(x, s^\beta) \quad (17)$$

Suppose that:

$$F_\alpha[\bar{u}(x, s^\beta); \omega^\alpha] = \bar{u}^*(\omega^\alpha, s^\beta), \quad F_\alpha[\bar{f}(x, s^\beta); \omega^\alpha] = \bar{f}^*(\omega^\alpha, s^\beta) \quad (18)$$

$$F_\alpha[\varphi(x); \omega^\alpha] = \varphi^*(\omega^\alpha, s^\beta), \quad F_\alpha[\psi(x); \omega^\alpha] = \psi^*(\omega^\alpha) \quad (19)$$

then, from (15) we get:

$$(i\omega)^{2\alpha} \bar{u}^*(\omega^\alpha, s^\beta) + s^{2\beta} \bar{u}^*(\omega^\alpha, s^\beta) - s^\beta \varphi^*(\omega^\alpha) - \psi^*(\omega^\alpha) + k^2 \bar{u}^*(\omega^\alpha, s^\beta) = \bar{f}^*(\omega^\alpha, s^\beta)$$

Therefore,

$$\bar{u}^*(\omega^\alpha, s^\beta) = \frac{s^\beta \varphi^*(\omega^\alpha) + \psi^*(\omega^\alpha) + \bar{f}^*(\omega^\alpha, s^\beta)}{s^{2\beta} + k^2 + (i\omega)^{2\alpha}} \quad (20)$$

Furthermore, by the inverse Yang-Fourier transform, we get the solution of the form:

$$\begin{aligned} u(x, y) = & \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) E_{2\beta, \beta}[-b(\omega^\alpha) y^{2\beta}] \varphi^*(\omega^\alpha) (d\omega)^\alpha + \\ & + \frac{1}{(2\pi)^\alpha} \int_{-\infty}^{+\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) y^\beta E_{2\beta, 2\beta}[-b(\omega^\alpha) y^{2\beta}] \psi^*(\omega^\alpha) (d\omega)^\alpha + \\ & + \frac{1}{(2\pi)^\alpha \Gamma(1 + \beta)} \int_{-\infty}^{+\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha) \int_0^y (y - \tau)^\beta E_{2\beta, 2\beta}[-b(y - \tau)^{2\beta}] f^*(\omega^\alpha, \tau) (d\tau)^\beta (d\omega)^\alpha \end{aligned}$$

Conclusion

In the present work, we considered a generalized local fractional 2-D Helmholtz equation. The Yang-Fourier transform and Yang-Laplace transform method were used to solve the equation. The integral expression of the solutions defined on the fractal sets was obtained. They are quite useful for engineers and scientists to analyze the behavior of the non-differentiable solution.

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Nomenclature

t – time co-ordinate, [s]
 x – space co-ordinate, [m]

Greek symbol
 α – fractal order, [-]

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