MATHEMATICAL MODEL OF ENERGY EFFICIENCY IN SPUR GEARS

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The impact of geometric parameters of teeth and lubricating oils to the efficiency of utilization of involute spur gears, when the transverse contact ratio is $2 < \varepsilon_\alpha \leq 3$*, has been analyzed in this paper. The mathematical model and computer program for determining the current and the effective value of the efficiency of utilization have been developed. The influence of the character of load distribution and energy losses due to heating effects during the meshing period is included in the factor of load distribution. The results of computer simulation are given in the form of a diagram of the current values of the efficiency of utilization during the meshing period. Also, the values of effective efficiency of utilization for the considered cylindrical gear pairs have been calculated.*

Key words*: spur gears, mesh teeth, load distribution, energy efficiency, energy losses, heating effects*

Introduction

Meshing profiles of gears teeth during the contact period is characterized by rolling and sliding, i.e. friction that occurs between flanks of teeth. Friction, as a negative occurrence, cannot be completely eliminated, which results in spending a part of power transmitted from the pinion to the gear on overcoming it. The power losses that occurs during the meshing of teeth is shown through efficiency of utilization. It is customary to quantify the energy efficiency of gear pairs by analyzing this efficiency of utilization. Given the increasingly stringent requirements in terms of energy conservation, the efficiency of utilization is a very important qualitative and quantitative characteristic in the selection of geometric parameters of gear pairs and an appropriate oil for their lubrication.

In order to increase the energy efficiency of the gear pairs in the literature [1-6] adequate theoretical and experimental research have been carried out. In the literature [1] an analytical model of load distribution in simultaneously meshed teeth pairs is developed, when during the contact period the single and double meshed teeth take turns. On the basis of the developed model, the impact of the load distribution on energy efficiency of gear pairs can be analyzed. In this literature is shown that load distribution for simultaneously meshed teeth pairs is primarily dependent on the compatibility of accuracy of making of teeth, their rigidity and load intensity of gear pairs. For the analysis of energy efficiency of planetary gearboxes the mathematical model for the case of uniform distribution of loads in simultaneously meshed teeth pairs is developed in the literature [2], when during the contact period the single and double mesh of teeth take turns. Some researchers focus on the determination of equations used for friction coefficient [7] computation, the rolling friction force and thickness of the oil film [8]. In many studies [9, 10] the influence of oil on the lubrication gear pairs is analyzed.

In this paper, a mathematical model for analyzing energy efficiency of cylindrical gear pairs with internal teeth is developed, when during the contact period the double and triple mesh of teeth take turns. Based on the developed model for determining efficiency of utilization all relevant parameters that have a distinct influence on the efficiency of utilization can be identified. The paper analyzes the impact of oil and position of the pitch point during the contact period on the energy losses due transformation in thermal energy and on the total efficiency of utilization of spur gear pairs with internal teeth. By varying the shape of the teeth profiles [5] the position of the pitch point during the contact period of meshed teeth was being changed.

Load distribution on the simultaneously meshed teeth pairs

For consideration of the load distribution on the simultaneously meshed teeth pairs, load distribution factor [1] is defined, which shows the level of engagement of certain simultaneously meshed teeth pairs in the transmission of the total load of gear pair:

$$K_{\alpha \mathrm{i}} = \frac{F_{\mathrm{ni}}}{F_{\mathrm{n}}} \tag{1}$$

where F_{ni} [N] – force transmitted by the considered teeth pair, F_{n} [N] – total force of gear pair, and i – number of simultaneously meshed teeth pairs.

From the aspect of load distribution, two characteristic extreme cases may appear with the simultaneously meshed teeth pairs. The first corresponds to the ideal even distribution of the load and the other a highly uneven distribution of the load, so that the actual distribution of the load is situated between these extreme cases. The actual load distribution depends on the elastic deformation of the simultaneously meshed teeth pairs, shafts and supports, and accuracy of making of teeth. In this load distribution, a set of simultaneously meshed teeth is considered as a statically indeterminate system. From the condition of elastic deformation and variations of the base pitch and the profiles shape of teeth one comes to an analytical expression for the load distribution factor [1].

At highly uneven distribution of the load it is characteristic that even when higher number of simultaneously meshed teeth pairs is present, total load is transmitted over a single teeth pair. Load distribution factor then reaches an upper extreme value ($K_{\alpha \mathrm{i}} = 1$). In the ideal even load distribution, all simultaneously meshed teeth pairs are equally involved in the transfer of the total load of the gear pair ($K_{\alpha \mathrm{i}} = 1/2$ with a double mesh, $K_{\alpha \mathrm{i}} = 1/3$ with a triple mesh, and so on), as shown in fig. 1.

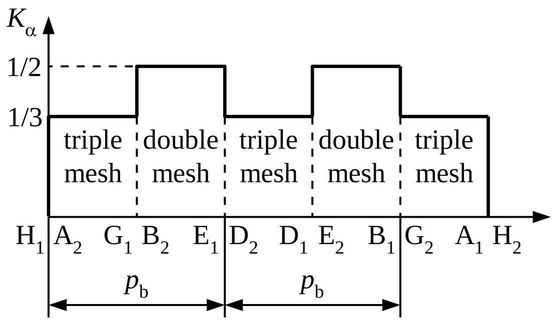

Figure 1. Points of contact at line of action of the profiles meshed teeth pairs in which the load handover is performed

Mathematical model for determination of the efficiency of utilization

Torque transferring T_1, from pinion to gear, is achieved by force by which the flank of the first gear affects the flank of the second gear. Wherein beside the normal force at the point of contact of meshed flanks of teeth, forces of sliding and rolling friction occur. Attacking lines of friction force lie in a common plane of meshed flanks and their directions change when the contact point passes through the pitch point.

Fig. 1 shows the points of contact on the action line of the teeth profiles in which handover of loads on the simultaneously meshed teeth pairs are carried out, i.e. points of contact in which double and triple mesh of teeth take turns. At the top of the tooth the point of contact A is located. It belongs to the addendum circle gear. Point of contact H is located at the tooth root. It is the last point of the profile tooth which participates in the transfer of the load. Between the points of contact A and H, there are contact points B, D, E and G, in which the load handover is performed on the simultaneously meshed teeth pairs.

Depending on the geometric parameters of gear pair, the pitch point may be in the area of double or triple mesh of the considered gear pair. It is therefore necessary to include the variable *IND* in the mathematical model which defines characteristic points in which the load handover is performed, compared to the pitch point. Fig. 2 shows the algorithm for determining the position of the characteristic points of contact in which the load handover is performed. In the presented algorithm, the variable of *IND* defines appropriate intervals of the load in comparison to the pitch point [2], so:

IND=1 – the pitch point is located in the triple mesh area, wherein the working pressure angle is higher than or equal to the angle in the point E_1 ($\alpha_{E1} \le \alpha_w$), and the working pressure angle is lower than the angle in the point D_1 ($\alpha_{D1} > \alpha_w$),

IND=2 – the pitch point is located in the double mesh area, wherein the working pressure angle is higher than or equal to the angle in the point D_1 ($\alpha_{D1} \le \alpha_w$), and the working pressure angle is lower than the angle in the point B_1 ($\alpha_{B1} > \alpha_w$),

IND=3 – the pitch point is located in the triple mesh area, wherein the working pressure angle is higher than or equal to the angle in the point B_1 ($\alpha_{B1} \le \alpha_w$), and the working pressure angle is lower than the angle in the point A_1 ($\alpha_{A1} > \alpha_w$),

IND=4 – the pitch point is located in the double mesh area, wherein the working pressure angle is higher than or equal to the angle in the point G_1 ($\alpha_{G1} \le \alpha_w$), and the working pressure angle is lower than the angle in the point E_1 ($\alpha_{E1} > \alpha_w$),

IND=5 – the pitch point is located in the triple mesh area, wherein the working pressure angle is higher than or equal to the angle in the point H_1 ($\alpha_{H1} \le \alpha_w$), and the working pressure angle is lower than the angle in the point G_1 ($\alpha_{G1} > \alpha_w$).

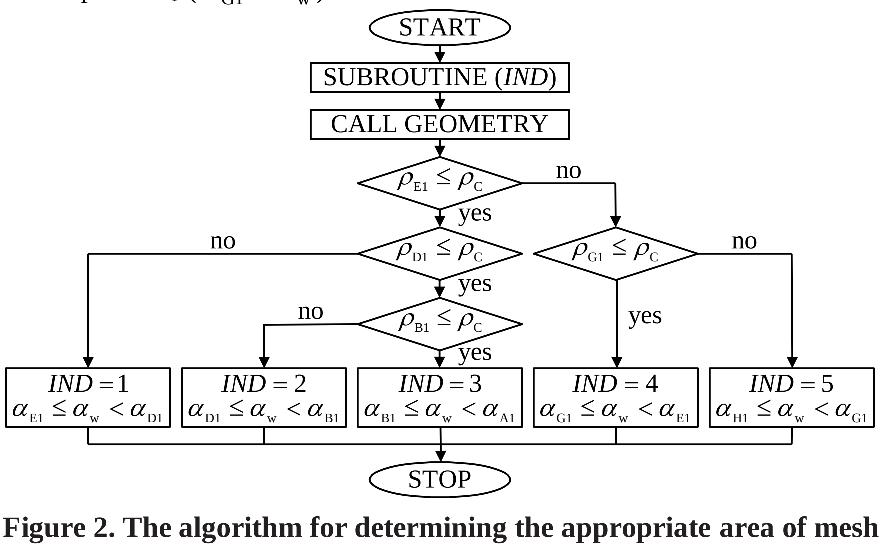

Figure 2. The algorithm for determining the appropriate area of mesh

To determine the efficiency of utilization of gear pairs, the following assumptions have been adopted:

– friction losses in the bearings are small in comparison to the losses due to sliding and rolling friction on the flanks of teeth meshed, which is, given the exclusive application of the rolling bearings with these transmitters, very close to reality,

– rotational speed, i.e. angular frequency of meshed gears is constant,

– torque acting on the working machine during operation of the gear unit power is constant.

The intensity of the sliding friction force, acting on the flanks of teeth meshed during the contact period depends on the normal force and the friction coefficient [6, 7] at the considered point of contact:

$$F_{\mu}(\xi) = \mu(\xi) \cdot F_{n} \tag{2}$$

$$\mu(\xi) = 0.0127 \cdot \log\left(\frac{\frac{29.66}{b} \cdot F_{n}}{\eta \cdot v_{s}(\xi) \cdot v_{r}^{2}(\xi)}\right) \tag{3}$$

where $\mu(\xi)$ – friction coefficient, b [m] – face width of gear pair, η [Pas] – dynamic viscosity of lubricating oil, $v_{s}(\xi)$[ms^{-2}] – sliding velocity of tooth profile of the pinion compared to the gear, and $v_{r}(\xi)$[ms^{-2}] – rolling velocity of tooth profile of the pinion compared to the gear.

The intensity of the rolling friction force, acting on the teeth flanks of meshed gears during the contact period, at a current point of contact can be displayed depending on the thickness of the oil film [8] separating the flanks of the meshed teeth of the gear pair:

$$F_{r}(\xi) = \mathrm{C} \cdot h(\xi) \cdot b \tag{4}$$

$$h(\xi) = 1.6 \cdot \alpha^{0.6} \cdot (\eta \cdot v_{r}(\xi))^{0.7} \cdot E^{0.003} \cdot \frac{R^{0.43}}{F_{n}^{0.13}} \tag{5}$$

where $\mathrm{C} = 9 \cdot 10^{7}$ – constant of proportionality, $h(\xi)$[m] – thickness of the oil film, α [Pa^{-1}] – pressure viscosity coefficient, R [m] – equivalent radius of curvature of meshed teeth profiles, and E [Nm^{-2}] – equivalent modul of elasticity.

Presuming that parameters of the gear pair are selected so that the pitch point (point C in fig. 3) is in the area of the double mesh, i.e. when *IND*=1. With that in mind, fig. 3 shows the forces acting on the simultaneously meshed teeth profiles in comparison to a fixed coordinate system attached to the axis of rotation of gear pair.

At the point of contact the normal force has a constant direction in comparison to the coordinate system attached to the axis of rotation of meshed gears, and friction forces are in the tangent plane of meshed flanks of teeth.

The intensity of the normal force, which acts on simultaneously meshed profiles of teeth is determined from the condition that the sum of torque to the pinion axis O_1 is equal to zero:

$$T_{1} - K_{\alpha 1} \cdot F_{n} \cdot \frac{3 \cdot d_{b1}}{2} + F'_{t1} \cdot \xi - F'_{t2} \cdot p_{1} - F'_{t3} \cdot p_{2} = 0 \tag{6}$$

$$F_{n} = \frac{T_{1} - F_{r1} \cdot \xi - F_{r2} \cdot p_{1} - F_{r3} \cdot p_{2}}{K_{\alpha 1} \cdot \frac{3 \cdot d_{b1}}{2} - \mu_{1} \cdot K_{\alpha 1} \cdot \xi + \mu_{2} \cdot K_{\alpha 2} \cdot p_{1} + \mu_{3} \cdot K_{\alpha 3} \cdot p_{2}} \tag{7}$$

where $\rho_{H1} \leq \xi < \rho_{G1}$ – area of the triple mesh, p_{b} [m] – the base pitch, $p_{1} = \xi + p_{b}$, $p_{2} = \xi + 2 \cdot p_{b}$, $F'_{t1} = \mu_{1} \cdot F_{n1} - F_{r1} = \mu_{1} \cdot K_{\alpha 1} \cdot F_{n} - F_{r1}$, $F''_{t1} = \mu_{1} \cdot F_{n1} + F_{r1} = \mu_{1} \cdot K_{\alpha 1} \cdot F_{n} + F_{r1}$,

$F'_{t2} = \mu_2 \cdot F_{n2} + F_{r2} = \mu_2 \cdot K_{\alpha 2} \cdot F_n + F_{r2}$, $\qquad F''_{t2} = \mu_2 \cdot F_{n2} - F_{r2} = \mu_2 \cdot K_{\alpha 2} \cdot F_n - F_{r2}$,

$F'_{t3} = \mu_3 \cdot F_{n3} + F_{r3} = \mu_3 \cdot K_{\alpha 3} \cdot F_n + F_{r3}$, $\qquad F''_{t3} = \mu_3 \cdot F_{n3} - F_{r3} = \mu_3 \cdot K_{\alpha 3} \cdot F_n - F_{r3}$,

and d_{b1}[m] – the base circle diameter of the pinion.

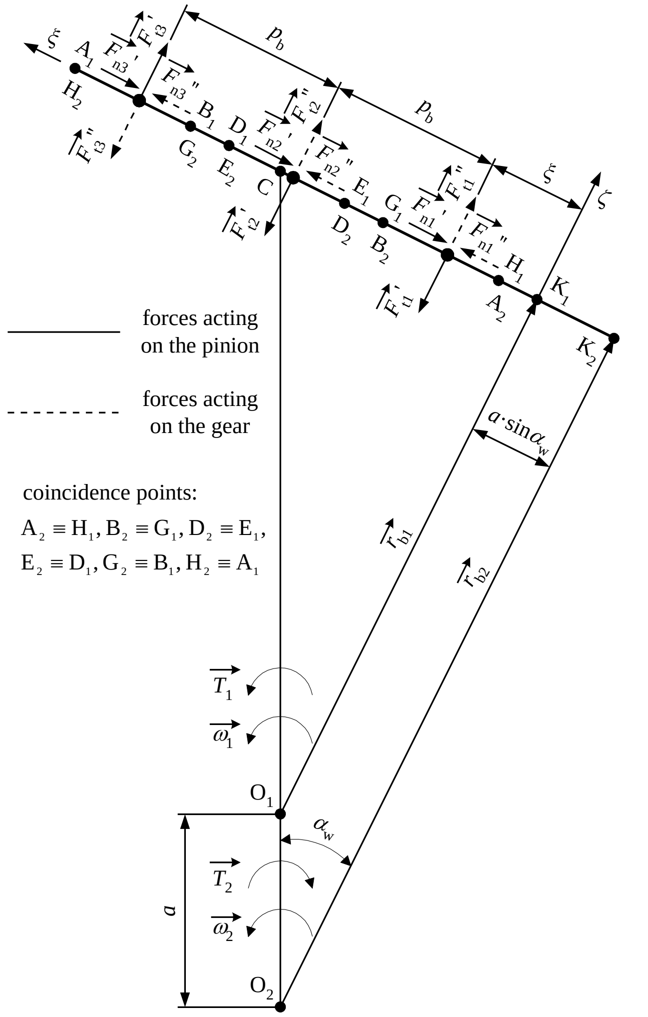

Figure 3. The load acting on the simultaneously meshed teeth profiles for *IND*=1

The intensity of the normal force cannot be explicitly determined based on the eq. (7), since the coefficient of friction and rolling friction force are the functions of the normal force. Therefore, it is necessary to assume an initial value for the intensity of the normal force which is in the first iteration:

$$F_{np1} = \frac{2 \cdot T_1}{d_{b1}} \tag{8}$$

Interactive method for determining the intensity of the normal force acting on the flanks of teeth of meshed gears, lasts until the requirement in terms of desired accuracy is met:

$$\left| \frac{F_{n(y+1)} - F_{n(y)}}{F_{n(y)}} \right| \leq \varepsilon \tag{9}$$

where $\varepsilon = 0.01$ – preset accuracy, and y – order number of iteration.

To determine the efficiency of utilization for a given gear pair, it is necessary to know the torque T_2 which acts on the gear. Starting from the condition that the sum of torque to the axis O_2 of the gear is zero, this torque is:

$$T_2 = K_{\alpha 2} \cdot F_n \cdot \frac{3 \cdot d_{b2}}{2} - F''_{t1} \cdot p_3 + F''_{t2} \cdot p_4 + F''_{t3} \cdot p_5 \tag{10}$$

where d_{b2} [m] – the base circle diameter of the gear, a [m] – center distance of the gear pair, α_w [°] – working pressure angle, $p_3 = a \cdot \sin\alpha_w + \xi$, $p_4 = a \cdot \sin\alpha_w + \xi + p_b$, and $p_5 = a \cdot \sin\alpha_w + \xi + 2 \cdot p_b$.

In the observed point of contact the torques acting on the gear pair are determined. Accordingly, the current values of the efficiency of utilization for a given gear pair in the observed point of contact can be determined:

$$\eta_\xi = \frac{T_2}{T_1} \cdot \frac{1}{u} \tag{11}$$

where T_1 [Nm] – torque acting on the pinion, T_2 [Nm] – torque acting on the gear, and u – gear ratio.

In a completely analogous way one determines the normal force and torque acting on the gear in the area of the double mesh $\rho_{G1} \le \xi < \rho_{E1}$. In doing so, the intensities of the normal force and torque acting on the gear are given by the following expressions:

$$F_n = \frac{T_1 - F_{r1} \cdot \xi - F_{r2} \cdot p_1}{K_{\alpha 1} \cdot d_{b1} - \mu_1 \cdot K_{\alpha 1} \cdot \xi + \mu_2 \cdot K_{\alpha 2} \cdot p_1} \tag{12}$$

$$T_2 = K_{\alpha 2} \cdot F_n \cdot d_{b2} - F''_{t1} \cdot p_3 + F''_{t2} \cdot p_4 \tag{13}$$

In a completely analogous way one determines the normal force and torque acting on the gear in the area of the triple mesh $\rho_{E1} \le \xi < \rho_C$. It should be noted here that the directions of sliding speed and sliding friction force change when the contact point goes through the pitch point. Therefore, the meshing teeth profiles is first considered in the area before the point of contact passes through the pitch point. In doing so, the intensities of the normal force and torque acting on the gear are given by the following expressions:

$$F_n = \frac{T_1 - F_{r1} \cdot p_6 + F_{r2} \cdot \xi - F_{r3} \cdot p_1}{K_{\alpha 1} \cdot \frac{3 \cdot d_{b1}}{2} - \mu_1 \cdot K_{\alpha 1} \cdot p_6 - \mu_2 \cdot K_{\alpha 2} \cdot \xi + \mu_3 \cdot K_{\alpha 3} \cdot p_1} \tag{14}$$

$$T_2 = K_{\alpha 2} \cdot F_n \cdot \frac{3 \cdot d_{b2}}{2} - F''_{t1} \cdot p_7 - F''_{t2} \cdot p_3 + F''_{t3} \cdot p_4 \tag{15}$$

where $p_6 = \xi - p_b$, and $p_7 = a \cdot \sin\alpha_w + \xi - p_b$.

If the meshing teeth profiles is examined in the area after the point of contact passes through the pitch point $\rho_C \le \xi < \rho_{D1}$, the intensities of the normal force and torque acting on the gear are:

$$F_n = \frac{T_1 - F_{r1} \cdot p_6 - F_{r2} \cdot \xi - F_{r3} \cdot p_1}{K_{\alpha 1} \cdot \frac{3 \cdot d_{b1}}{2} - \mu_1 \cdot K_{\alpha 1} \cdot p_6 + \mu_2 \cdot K_{\alpha 2} \cdot \xi + \mu_3 \cdot K_{\alpha 3} \cdot p_1} \tag{16}$$

$$T_2 = K_{\alpha 2} \cdot F_n \cdot \frac{3 \cdot d_{b2}}{2} - F''_{t1} \cdot p_7 + F''_{t2} \cdot p_3 + F''_{t3} \cdot p_4 \tag{17}$$

If the meshing teeth profiles is considered after the point of contact is out of the area of the triple mesh and re-entring into the area of the double mesh $\rho_{D1} \le \xi < \rho_{B1}$, the intensities of the normal force and torque acting on the gear are:

$$F_n = \frac{T_1 - F_{r1} \cdot p_6 - F_{r2} \cdot \xi}{K_{\alpha 1} \cdot d_{b1} - \mu_1 \cdot K_{\alpha 1} \cdot p_6 + \mu_2 \cdot K_{\alpha 2} \cdot \xi} \tag{18}$$

$$T_2 = K_{\alpha 2} \cdot F_n \cdot d_{b2} - F''_{t1} \cdot p_7 + F''_{t2} \cdot p_3 \tag{19}$$

On the other side, when the contact of meshed teeth profiles is out of the area of the double mesh and re-entring into the area of the triple mesh $\rho_{B1} \le \xi < \rho_{A1}$, these variables are:

$$F_{\mathrm{n}}=\frac{T_1-F_{\mathrm{r1}}\cdot p_8-F_{\mathrm{r2}}\cdot p_6-F_{\mathrm{r3}}\cdot\xi}{K_{\alpha1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2}-\mu_1\cdot K_{\alpha1}\cdot p_8+\mu_2\cdot K_{\alpha2}\cdot p_6+\mu_3\cdot K_{\alpha3}\cdot\xi}\tag{20}$$

$$T_2=K_{\alpha2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2}-F''_{\mathrm{t1}}\cdot p_9+F''_{\mathrm{t2}}\cdot p_7+F''_{\mathrm{t3}}\cdot p_3\tag{21}$$

where $p_8=\xi-2\cdot p_{\mathrm{b}}$, and $p_9=a\cdot\sin\alpha_{\mathrm{w}}+\xi-2\cdot p_{\mathrm{b}}$.

Therefore, an effective efficiency of utilization for a given gear pair may be determined based on the following relation:

$$\eta_{\mathrm{ef}}=\frac{1}{l}\cdot\int_{\mathrm{H_1}}^{\mathrm{A_1}}\eta_{\xi}\cdot d\xi=\frac{1}{l}\cdot\int_{\mathrm{H_1}}^{\mathrm{A_1}}\frac{T_2}{T_1}\cdot\frac{1}{u}\cdot d\xi=\frac{1}{l\cdot T_1\cdot u}\cdot\int_{\mathrm{H_1}}^{\mathrm{A_1}}T_2\cdot d\xi\tag{22}$$

where l [m] – length of action line.

It is necessary to consider the scenario of meshing when the gear pair parameters are chosen so that the pitch point is in the area of double mesh, that is between points $\mathrm{D_1}$ and $\mathrm{B_1}$. In a completely analogous manner, as in the previous case, one can determine the normal force F_{n} and torque T_2 for *IND*=2. In tab. 1 are given the final expressions for the normal force F_{n} and torque T_2 for *IND*=2.

Table 1. Normal force F_{n} and torque T_2 for *IND*=2

IND=2			F_{n} [N]
			T_2 [Nm]
Mesh area	Triple	$\rho_{\mathrm{H1}}\le\xi<\rho_{\mathrm{G1}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot\xi-F_{\mathrm{r2}}\cdot p_1-F_{\mathrm{r3}}\cdot p_2}{K_{\alpha1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2}-\mu_1\cdot K_{\alpha1}\cdot\xi+\mu_2\cdot K_{\alpha2}\cdot p_1+\mu_3\cdot K_{\alpha3}\cdot p_2}$
			$K_{\alpha2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2}-F''_{\mathrm{t1}}\cdot p_3+F''_{\mathrm{t2}}\cdot p_4+F''_{\mathrm{t3}}\cdot p_5$
	Double	$\rho_{\mathrm{G1}}\le\xi<\rho_{\mathrm{E1}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot\xi-F_{\mathrm{r2}}\cdot p_1}{K_{\alpha1}\cdot d_{\mathrm{b1}}-\mu_1\cdot K_{\alpha1}\cdot\xi+\mu_2\cdot K_{\alpha2}\cdot p_1}$
			$K_{\alpha2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}}-F''_{\mathrm{t1}}\cdot p_3+F''_{\mathrm{t2}}\cdot p_4$
	Triple	$\rho_{\mathrm{E1}}\le\xi<\rho_{\mathrm{D1}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot p_6-F_{\mathrm{r2}}\cdot\xi-F_{\mathrm{r3}}\cdot p_1}{K_{\alpha1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2}-\mu_1\cdot K_{\alpha1}\cdot p_6+\mu_2\cdot K_{\alpha2}\cdot\xi+\mu_3\cdot K_{\alpha3}\cdot p_1}$
			$K_{\alpha2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2}-F''_{\mathrm{t1}}\cdot p_7+F''_{\mathrm{t2}}\cdot p_3+F''_{\mathrm{t3}}\cdot p_4$
	Double	$\rho_{\mathrm{D1}}\le\xi<\rho_{\mathrm{C}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot p_6+F_{\mathrm{r2}}\cdot\xi}{K_{\alpha1}\cdot d_{\mathrm{b1}}-\mu_1\cdot K_{\alpha1}\cdot p_6-\mu_2\cdot K_{\alpha2}\cdot\xi}$
			$K_{\alpha2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}}-F''_{\mathrm{t1}}\cdot p_7-F''_{\mathrm{t2}}\cdot p_3$
		$\rho_{\mathrm{C}}\le\xi<\rho_{\mathrm{B1}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot p_6-F_{\mathrm{r2}}\cdot\xi}{K_{\alpha1}\cdot d_{\mathrm{b1}}-\mu_1\cdot K_{\alpha1}\cdot p_6+\mu_2\cdot K_{\alpha2}\cdot\xi}$
			$K_{\alpha2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}}-F''_{\mathrm{t1}}\cdot p_7+F''_{\mathrm{t2}}\cdot p_3$
	Triple	$\rho_{\mathrm{B1}}\le\xi<\rho_{\mathrm{A1}}$	$\frac{T_1-F_{\mathrm{r1}}\cdot p_8-F_{\mathrm{r2}}\cdot p_6-F_{\mathrm{r3}}\cdot\xi}{K_{\alpha1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2}-\mu_1\cdot K_{\alpha1}\cdot p_8+\mu_2\cdot K_{\alpha2}\cdot p_6+\mu_3\cdot K_{\alpha3}\cdot\xi}$

			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot \frac{3\cdot d_{\mathrm{b2}}}{2} - F''_{\mathrm{t1}}\cdot p_9 + F''_{\mathrm{t2}}\cdot p_7 + F''_{\mathrm{t3}}\cdot p_3$

The next scenario is for case when the gear pair parameters are chosen so that the pitch point is in the area of double mesh, that is between points G_1 and E_1. In a completely analogous manner, as for the two previous cases, one can determine the normal force F_{n} and torque T_2 for *IND*=4. In tab. 2 are given the final expressions for the normal force F_{n} and torque T_2 for *IND*=4.

Table 2. Normal force F_{n} and torque T_2 for *IND*=4

IND=4			F_{n} [N]
			T_2 [Nm]
Mesh area	Triple	$\rho_{\mathrm{H1}} \le \xi < \rho_{\mathrm{G1}}$	$\dfrac{T_1 - F_{\mathrm{r1}}\cdot\xi - F_{\mathrm{r2}}\cdot p_1 - F_{\mathrm{r3}}\cdot p_2}{K_{\alpha 1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2} - \mu_1\cdot K_{\alpha 1}\cdot\xi + \mu_2\cdot K_{\alpha 2}\cdot p_1 + \mu_3\cdot K_{\alpha 3}\cdot p_2}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2} - F''_{\mathrm{t1}}\cdot p_3 + F''_{\mathrm{t2}}\cdot p_4 + F''_{\mathrm{t3}}\cdot p_5$
	Double	$\rho_{\mathrm{G1}} \le \xi < \rho_{\mathrm{C}}$	$\dfrac{T_1 - F_{\mathrm{r1}}\cdot\xi - F_{\mathrm{r2}}\cdot p_1}{K_{\alpha 1}\cdot d_{\mathrm{b1}} - \mu_1\cdot K_{\alpha 1}\cdot\xi + \mu_2\cdot K_{\alpha 2}\cdot p_1}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}} - F''_{\mathrm{t1}}\cdot p_3 + F''_{\mathrm{t2}}\cdot p_4$
		$\rho_{\mathrm{C}} \le \xi < \rho_{\mathrm{E1}}$	$\dfrac{T_1 + F_{\mathrm{r1}}\cdot\xi - F_{\mathrm{r2}}\cdot p_1}{K_{\alpha 1}\cdot d_{\mathrm{b1}} + \mu_1\cdot K_{\alpha 1}\cdot\xi + \mu_2\cdot K_{\alpha 2}\cdot p_1}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}} + F''_{\mathrm{t1}}\cdot p_3 + F''_{\mathrm{t2}}\cdot p_4$
	Triple	$\rho_{\mathrm{E1}} \le \xi < \rho_{\mathrm{D1}}$	$\dfrac{T_1 - F_{\mathrm{r1}}\cdot p_6 - F_{\mathrm{r2}}\cdot\xi - F_{\mathrm{r3}}\cdot p_1}{K_{\alpha 1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2} - \mu_1\cdot K_{\alpha 1}\cdot p_6 + \mu_2\cdot K_{\alpha 2}\cdot\xi + \mu_3\cdot K_{\alpha 3}\cdot p_1}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2} - F''_{\mathrm{t1}}\cdot p_7 + F''_{\mathrm{t2}}\cdot p_3 + F''_{\mathrm{t3}}\cdot p_4$
	Double	$\rho_{\mathrm{D1}} \le \xi < \rho_{\mathrm{B1}}$	$\dfrac{T_1 - F_{\mathrm{r1}}\cdot p_6 - F_{\mathrm{r2}}\cdot\xi}{K_{\alpha 1}\cdot d_{\mathrm{b1}} - \mu_1\cdot K_{\alpha 1}\cdot p_6 + \mu_2\cdot K_{\alpha 2}\cdot\xi}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot d_{\mathrm{b2}} - F''_{\mathrm{t1}}\cdot p_7 + F''_{\mathrm{t2}}\cdot p_3$
	Triple	$\rho_{\mathrm{B1}} \le \xi < \rho_{\mathrm{A1}}$	$\dfrac{T_1 - F_{\mathrm{r1}}\cdot p_8 - F_{\mathrm{r2}}\cdot p_6 - F_{\mathrm{r3}}\cdot\xi}{K_{\alpha 1}\cdot\frac{3\cdot d_{\mathrm{b1}}}{2} - \mu_1\cdot K_{\alpha 1}\cdot p_8 + \mu_2\cdot K_{\alpha 2}\cdot p_6 + \mu_3\cdot K_{\alpha 3}\cdot\xi}$
			$K_{\alpha 2}\cdot F_{\mathrm{n}}\cdot\frac{3\cdot d_{\mathrm{b2}}}{2} - F''_{\mathrm{t1}}\cdot p_9 + F''_{\mathrm{t2}}\cdot p_7 + F''_{\mathrm{t3}}\cdot p_3$

Cases *IND*=3 and *IND*=5 are theoretically possible, but they are almost unachievable in practice due to the geometry of meshed teeth pairs. Therefore, they are not taken into consideration in this presentation.

Results

Results of research conducted in this paper are shown in diagram form in figs. from 4 to 9. The flow of changes of the current value of efficiency of utilization of spur gear pairs during the contact period is shown in these diagrams for two types of oil. All diagrams are developed for internal gear pairs with the rotational frequency of the pinion $n_1 = 1000\,1/\mathrm{min}$, torque on the pinion $T_1 = 150\,\mathrm{Nm}$, number of teeth of the pinion $z_1 = 50$, number of teeth of the gear $z_2 = 200$, standard module $m_n = 4\,\mathrm{mm}$, face width gears $b = 50\,\mathrm{mm}$ and pressure angle $\alpha_0 = 20°$. The shape of the profiles of simultaneously meshed teeth pairs was varied by selecting the appropriate shifting coefficient of teeth profiles [5], tab. 3.

The values of effective efficiency of utilization are in the range of 0.9820 to 0.9875 for the first oil [9] (figs. 4, 6 and 8) and in the range of 0.9889 to 0.9918 for the second oil [10] (figs. 5, 7 and 9), tab. 4. The energy efficiency of all shown gear pairs are in the range of 98.20% to 99.18%.

Table 3. Transverse contact ratio ε_α, shifting coefficient of teeth profiles of the pinion x_1 and gear x_2

IND	ε_α [–]	x_1 [–]	x_2 [–]
1	2.78	– 0.3	0.3
2	2.16	– 0.4	– 0.3
4	2.53	0	0.5

Table 4. Dynamic viscosity η, pressure viscosity coefficient α and ambient temperature T_0

Oil	η [Pas]	α [Pa^{-1}]	T_0 [°C]
first	0.075	$2.190 \cdot 10^{-8}$	40
second	0.036	$1.344 \cdot 10^{-8}$	60

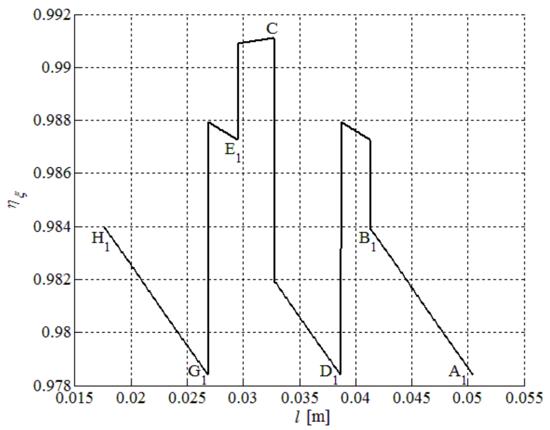

Figure 4. Diagram of change of the current value of the efficiency of utilization for *IND*=1 and first oil ($\eta_{\mathrm{ef}}^{\mathrm{f}} = 0.9820$)

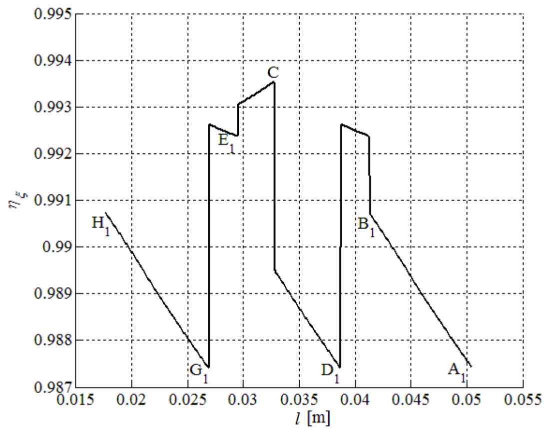

Figure 5. Diagram of change of the current value of the efficiency of utilization for *IND*=1 and second oil ($\eta_{\mathrm{ef}}^{\mathrm{s}} = 0.9889$)

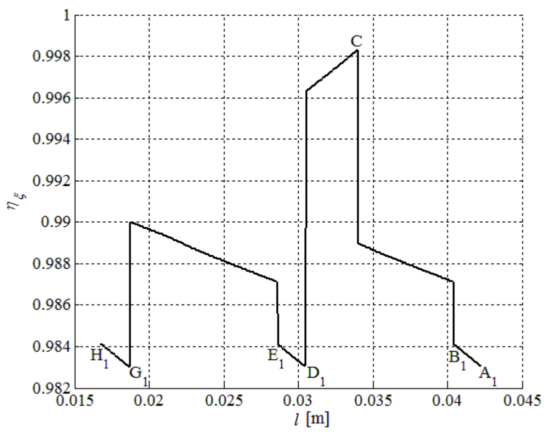

Figure 6. Diagram of change of the current value of the efficiency of utilization for *IND*=2 and first oil ($\eta_{\mathrm{ef}}^{\mathrm{f}} = 0.9875$)

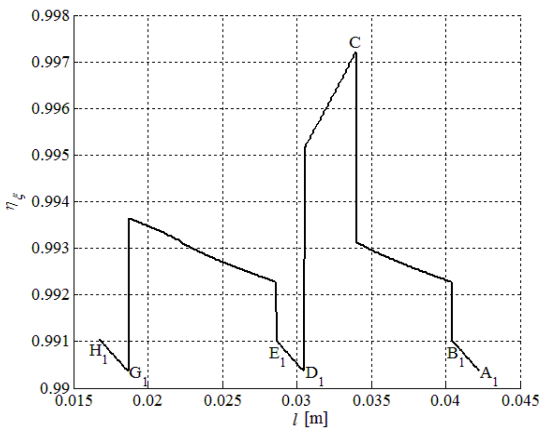

Figure 7. Diagram of change of the current value of the efficiency of utilization for *IND*=2 and second oil ($\eta_{ef}^{s} = 0.9918$)

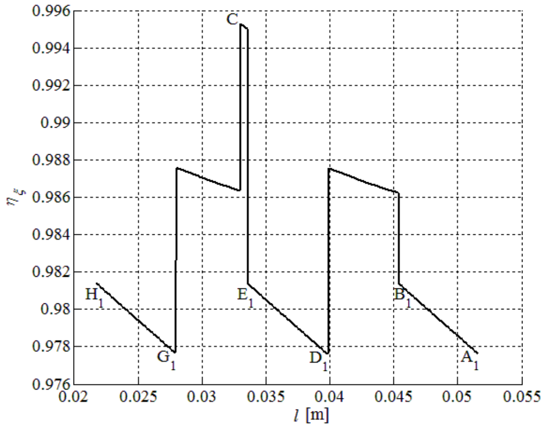

Figure 8. Diagram of change of the current value of the efficiency of utilization for *IND*=4 and first oil ($\eta_{ef}^{f} = 0.9821$)

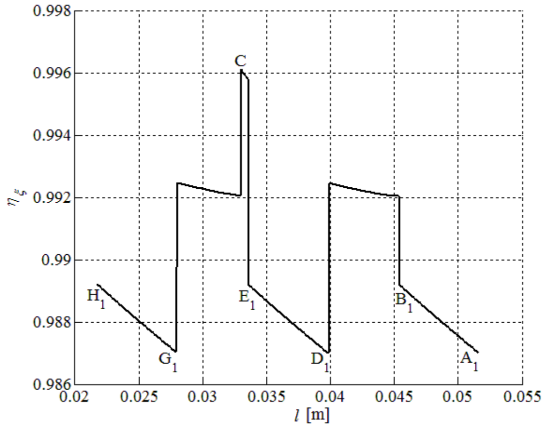

Figure 9. Diagram of change of the current value of the efficiency of utilization for *IND*=4 and second oil ($\eta_{ef}^{s} = 0.9893$)

By comparing the diagrams and results of these six combinations (figs. from 4 to 9), it follows that the highest values of the current and effective efficiency of utilization are achieved in gear pair in the case when is *IND*=2, tab. 3. In this gear pair, lubricated with second oil, tab. 4, the pitch point is located in the area of the double mesh between points D_1 and B_1. The energy efficiency of this combination of gear pair and oil is 99.18% due to the low transverse contact ratio this of gear pair, as it shown also in literature [10], as well as characteristics of the considered oil. Bearing in mind the fact that the shear stress between the oil layers is proportional to the oil viscosity, higher efficiency of utilization were obtained using the second oil in all three variants of gear pairs.

The research of influence of geometric and kinematic values of the gear pairs and the oil viscosity on energy losses ($\Delta\eta_{\xi} = 1 - \eta_{\xi}$) are shown in the diagrams in fig. 10. The diagrams show the ratio between the average values of energy losses due transformation in thermal energy in the area of double and triple mesh for considered oils (first-f and second-s) and gear pairs, tabs. 3 and 4. Energy losses appear as result of overcoming the resistance that occurs in the oil as well as due to friction on the flanks of the simultaneously meshed teeth pairs.

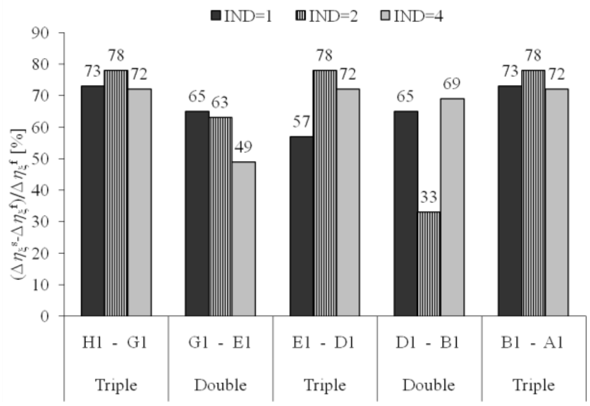

Figure 10. The ratio of average values of energy losses due transformation in thermal energy **per mesh area for considered oils and gear pairs**

Based on obtained results (fig. 10), it is shown that the lowest difference in energy losses is generated in the area of meshing of the teeth flanks where the rolling friction is dominant (e.g. 33% for *IND*=2). In this area is located the pitch point, i.e. the point in which the sliding velocity of teeth flanks is zero. The largest difference in energy losses is found in the areas of the triple mesh (e.g. 78% for *IND*=2), when pitch point is not located in these areas. Contrary to the areas of the triple mesh, in the areas of the double mesh lower difference in energy losses are obtained (e.g. 63% for *IND*=2) because of smaller number of contact surfaces.

Simultaneous influence of teeth geometry, load, oil viscosity, sliding and rolling velocity on the energy efficiency of cylindrical gear pairs with straight teeth is shown in the diagrams in figs. from 11 to 13. The two previously indicated oils were considered [9, 10], tab. 4. The teeth geometry is changed by selecting appropriate shifting coefficient of teeth profiles [5], tab. 3. Torque on the pinion T_1 varied in the interval 50-250 Nm. The impact of sliding and rolling velocity was considered through the change of the rotational frequency of the pinion n_1 in the interval 500-1500 $1/\mathrm{min}$.

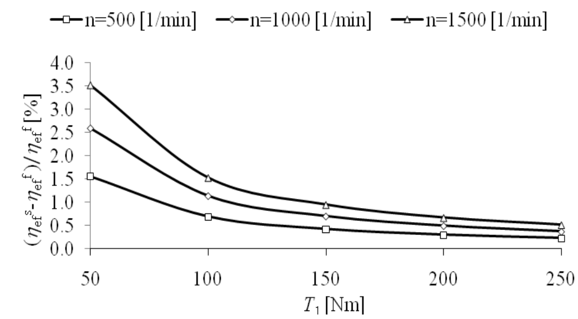

Figure 11. The ratio of effective efficiency of utilization for considered oils and *IND*=1

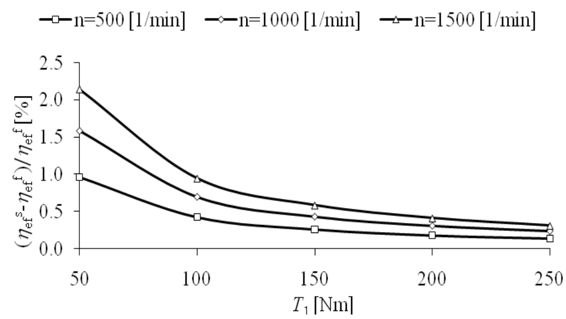

Figure 12. The ratio of effective efficiency of utilization for considered oils and *IND*=2

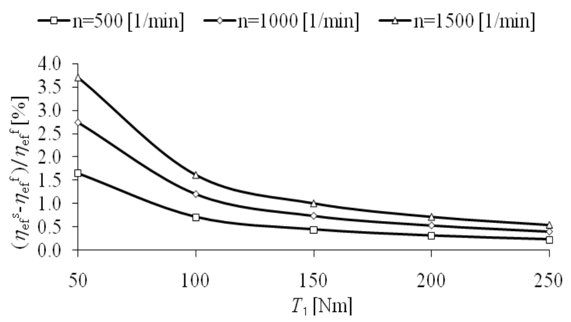

Figure 13. The ratio of effective efficiency of utilization for considered oils and *IND*=4

Based on the obtained results shown in diagrams in figs. from 11 to 13 it can be concluded that when oil has lower dynamic viscosity and pressure viscosity coefficient the considered gear pairs have higher energy efficiency (from minimum of 0.14% for *IND*=2, $T_1 = 250\,\mathrm{Nm}$ and $n_1 = 500\,1/\mathrm{min}$ to maximum of 3.71% for *IND*=4, $T_1 = 50\,\mathrm{Nm}$ and $n_1 = 1500\,1/\mathrm{min}$, figs. 12 and 13) compared to the oil which has higher dynamic viscosity and pressure viscosity coefficient in the considered range of load and rotational frequency. Thereby, with the increase of torque this difference in the energy efficiency for the considered oils decreases (e.g. from 3.51% to 0.52% for *IND*=1 and $n_1 = 1500\,1/\mathrm{min}$, fig. 11) and it decreases much more in the area of the high intensity of the load. However, with higher rotational frequency this difference in the energy efficiency for the considered oils increases (e.g. from 1.65% to 3.71% for *IND*=4 and $T_1 = 50\,\mathrm{Nm}$, fig. 13). From the aspect of teeth geometry the best energy efficiency is achieved in gear pair for the case when *IND*=2, because it has the lowest difference in the energy efficiency for the considered oils (from minimum of 0.14% for $T_1 = 250\,\mathrm{Nm}$ and $n_1 = 500\,1/\mathrm{min}$ to maximum of 2.15% for $T_1 = 50\,\mathrm{Nm}$ and $n_1 = 1500\,1/\mathrm{min}$, fig. 12).

Conclusions

The graphic presentation of the results obtained shows that changes of the efficiency of utilization follow the character of changes of load distribution on the simultaneously meshed teeth pairs. Based on the received results it follows that the current value of the efficiency of utilization is maximal in the pitch point. Higher values of the efficiency of utilization can be found in the area of the double mesh, and lower values in the area of the triple mesh. Energy losses caused by the rolling friction is much lower than the energy losses caused by sliding friction.

The paper shows that when oil has lower dynamic viscosity and pressure viscosity coefficient the considered gear pairs have higher energy efficiency compared to the oil which has higher dynamic viscosity and pressure viscosity coefficient in the considered range of load and rotational frequency. With the increase of torque this difference in the energy efficiency of the considered oils decreases, and with higher rotational frequency this difference increases.

While energy losses due its transformation in thermal energy of single gear pair are quite small, energy efficiency of gearbox that consists of few gear pairs may be significant. The total energy efficiency of gearbox formed of few gear pairs is product of energy efficiency of every single gear pair. Accordingly, any improvement in energy efficiency that can be achieved in each gear pair significantly contribute to energy efficiency of entire gearbox.

The developed mathematical model in this paper and the results of research obtained can be used in designing energy efficient of gearboxes. Also, it can be used for further research in order to form more complex models to analyze the energy efficiency of gearboxes.

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