N-WAVE AND OTHER SOLUTIONS TO THE B-TYPE KADOMTSEV–PETVIASHVILI EQUATION

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The present article studies a B-type Kadomtsev–Petviashvili (KP) equation with certain applications in the fluids. Stating with the Hirota’s bilinear form and adopting reliable methodologies, a group of exact solutions such as the N-wave and other solutions to the B-type KP equation is formally derived. Some figures in two and three dimensions are given to illustrate the characteristics of the obtained solutions. The results of the current work actually help to complete the previous studies about the B-type KP equation.

Key words: B-type Kadomtsev–Petviashvili equation, Hirota’s bilinear form, reliable methodologies, N-wave and other solutions

Introduction

During the last two decades, the topic of numerous studies has been about the integrable equations. Many researchers have focused on studying the integrable equations and their exact solutions. Because, the integrable equations describe many real phenomena in the broad branches of science and engineering. There is a wide range of reliable methods that can be used to handle the integrable equations; for instance, Kudryashov methods [1-6], ansatz methods [7-11], simplified Hirota’s method [12-16], linear superposition method [17-19], and multiple exp-function method [20-22].
The B-type KP equations are considered as nonlinear models in the fluids or the plasmas which have been investigated using different methods [23-37]. In this paper, a B-type KP equation with certain applications in the fluids is studied such that its mathematical model can be expressed as [36, 37]

$$
\frac{\partial}{\partial x} \left( \frac{\partial^5 u(x,y,t)}{\partial x^5} \right) + 60 \left( \frac{\partial u(x,y,t)}{\partial x} \right)^3 + 5 \frac{\partial^3 u(x,y,t)}{\partial x^3} \frac{\partial u(x,y,t)}{\partial y} + 30 \frac{\partial^3 u(x,y,t)}{\partial x^3} \frac{\partial u(x,y,t)}{\partial x} + 30 \frac{\partial u(x,y,t)}{\partial x} \frac{\partial u(x,y,t)}{\partial t} - 5 \frac{\partial^2 u(x,y,t)}{\partial t^2} = 0.
$$

(1)

The B-type KP equation (1) passes the Painlevé test and in the context of Painlevé feature and Hirota’s formalism is an integrable equation [36]. Such an integrable equation has been investigated by reliable methods. Singh and Gupta [36] obtained the soliton solutions and Du et al. [37] derived the lump and other wave solutions of the B-type KP equation. The reader can see [38-50].

The Hirota’s bilinear form of the B-type KP equation is given as [36, 37]

$$(D_x^6 + 5D_x^2D_t - 5D_t^2 + D_xD_y)f \cdot f = 0, \quad u = (\ln f)_x,$$

(2)

where $f$ is an unknown function and $D$ is the Hirota’s operator. The organization of this article is as follows: in Section 2, the linear superposition method is adopted to gain the $N$-wave and complexiton solutions of the B-type KP equation. In Section 3, a series of ansatz methods is exerted to derive other solutions of the B-type KP equation. The last section summarizes the findings of the present paper.

**N-wave and other solutions**

To acquire the $N$-wave solutions of the B-type KP equation, we search $a_j, j = 1,2,3$ and $c_j, j = 1,2,3$ such that

$$a_1^6(x^{c_1} - y^{c_1})^6 + 5a_1^3(x^{c_1} - y^{c_1})^3 a_3(x^{c_3} - y^{c_3}) - 5a_3^2(x^{c_3} - y^{c_3})^2 + a_1(x^{c_1} - y^{c_1})a_2(x^{c_2} - y^{c_2}) = 0.
$$

(3)

It is easy to show that $a_j, j = 1,2,3$ can be obtained if $c_1 = 1$, $c_2 = 5$, and $c_3 = 3$. By inserting $c_j, j = 1,2,3$ into Eq. (3) and using some operations, one can obtain

$$a_1^6 + 5a_1^3a_3 - 5a_3^2 + a_4a_2 = 0,$$

$$- 6a_1^6 - 15a_1^3a_3 - a_4a_2 = 0,$$

$$a_1^6 + a_3^2 = 0,$$

$$- 2a_1^6 - a_3^2a_3 + a_3^2 = 0.
$$

The above nonlinear system can be solved for deriving
$a_2 = 9a_1^5$, $a_3 = -a_1^3$.

Now, the following $N$-wave, complexiton, and positive complexiton solutions to the equation (1) can be constructed

\[ u = (\ln f)_x, \quad f = \sum_{j=1}^N d_j e^{\theta_j}, \quad \theta_j = k_j x + 9k_j^5 y - k_j^3 t; \]

\[ u = (\ln f)_x, \quad f = \sum_{j=1}^N e^{\theta_{j,1}} \left( d_{j,1} \cos \left( \theta_{j,2} \right) + d_{j,2} \sin \left( \theta_{j,2} \right) \right), \quad \theta_j = k_j x + 9k_j^5 y - k_j^3 t = \theta_{j,1} + i\theta_{j,2}, \quad d_{j,1}, d_{j,2} \in \mathbb{R}, \quad i^2 = -1; \]

\[ u = (\ln f)_x, \quad f = \sum_{j=1}^N d_j \cosh \left( k_j x + 9k_j^5 y - k_j^3 t \right) + \sum_{j=N+1}^{N+M} d_j \cos \left( k_j x + 9k_j^5 y + k_j^3 t \right), \quad d_j > 0 \text{ for } j = 1, 2, \ldots, N \text{ and } \sum_{j=1}^N d_j > \sum_{j=N+1}^{N+M} |d_j|. \]

Particularly, if we select $N = M = 1$, then the positive complexiton solution can be written as

\[ u = (\ln f)_x, \quad (4) \]

in which

\[ f = d_1 \cosh \left( k_1 x + 9k_1^5 y - k_1^3 t \right) + d_2 \cos \left( k_2 x + 9k_2^5 y + k_2^3 t \right), \quad d_1, d_2 > 0, \quad d_1 > |d_2|. \]

Figure 1 shows the positive complexiton solution (4) on the x-y plane when $d_1 = 1.5$, $d_2 = 1$, $k_1 = 1$, $k_2 = 2.5$, and $t = 0$.

Figure 1. The positive complexiton solution (4) on the x-y plane when $d_1 = 1.5$, $d_2 = 1$, $k_1 = 1$, $k_2 = 2.5$, and $t = 0$.

Other rational solutions

In the current section, a series of ansatz methods are formally utilized to derive other rational solutions of the B-type KP equation.

**Rational tanh method**

Let us consider the solution of the B-type KP equation as
\[ u(x, y, t) = \frac{\tanh(\kappa x + \eta y - \omega t)}{\rho + \sigma \tanh(\kappa x + \eta y - \omega t)} \]  \tag{5}

By setting Eq. (5) in Eq. (1) and using a number of operations, we find

\[ \kappa = \frac{\rho}{2(\rho - \sigma)(\rho + \sigma)}, \quad \tau = \frac{9 \rho^5}{2(\rho - \sigma)^5(\rho + \sigma)^3}, \quad \omega = \frac{\rho^3}{2(\rho - \sigma)^3(\rho + \sigma)^3}. \]

Now, a rational solution to the equation (1) can be constructed as

\[ u(x, y, t) = \frac{\tanh\left(\frac{\rho}{2(\rho - \sigma)(\rho + \sigma)} x + \frac{9 \rho^5}{2(\rho - \sigma)^5(\rho + \sigma)^3} y - \frac{\rho^3}{2(\rho - \sigma)^3(\rho + \sigma)^3} t \right)}{\rho + \sigma \tanh\left(\frac{\rho}{2(\rho - \sigma)(\rho + \sigma)} x + \frac{9 \rho^5}{2(\rho - \sigma)^5(\rho + \sigma)^3} y - \frac{\rho^3}{2(\rho - \sigma)^3(\rho + \sigma)^3} t \right)} \]  \tag{6}

It is easy to show that

\[ u(x, y, t) = \frac{\coth\left(\frac{\rho}{2(\rho - \sigma)(\rho + \sigma)} x + \frac{9 \rho^5}{2(\rho - \sigma)^5(\rho + \sigma)^3} y - \frac{\rho^3}{2(\rho - \sigma)^3(\rho + \sigma)^3} t \right)}{\rho + \sigma \coth\left(\frac{\rho}{2(\rho - \sigma)(\rho + \sigma)} x + \frac{9 \rho^5}{2(\rho - \sigma)^5(\rho + \sigma)^3} y - \frac{\rho^3}{2(\rho - \sigma)^3(\rho + \sigma)^3} t \right)} \]

is another rational solution of equation (1). Figure 2 illustrates the rational solution (6) on the x-y plane when \( \rho = -1, \sigma = -5, \) and \( t = 0. \)

![Figure 2](image_url)

**Figure 2.** The rational solution (6) on the x-y plane when \( \rho = -1, \sigma = -5, \) and \( t = 0. \)

**Rational cosh-sinh method**

The rational cosh-sinh approach considers the solution of the B-type KP equation as

\[ u(x, y, t) = \frac{\rho \cosh(\kappa x + \eta y - \omega t)}{\sigma \cosh(\kappa x + \eta y - \omega t) + \psi \sinh(\kappa x + \eta y - \omega t)} \]  \tag{7}

As before, by setting Eq. (7) in Eq. (1) and using a number of operations, we acquire

\[ \rho = \frac{2(\sigma^2 + \eta^2)}{\psi}, \quad \tau = 144 \kappa^5, \quad \omega = 4 \kappa^3. \]

Now, the following rational solution to the equation (1) can be derived
\[
 u(x, y, t) = \frac{2\psi(-\sigma^2 + \psi^2) \cosh(kx + 144k^5y - 4\kappa^3t)}{\psi(\sigma \cosh(kx + 144k^5y - 4\kappa^3t) + \psi \sinh(kx + 144k^5y - 4\kappa^3t))}.
\]

**Rational tan method**

Starting with

\[
 u(x, y, t) = \frac{\tan(kx + ry - \omega t)}{\rho + \sigma \tan(kx + ry - \omega t)},
\]

as the solution of the B-type KP equation, substituting it into Eq. (1), and using a number of operations, we gain

\[
 \kappa = -\frac{\rho}{2(\rho^2 + \sigma^2)}, \quad \tau = -\frac{9\rho^5}{2(\rho^2 + \sigma^2)^3}, \quad \omega = \frac{\rho^3}{2(\rho^2 + \sigma^2)^3}.
\]

Now, a rational solution to the equation (1) can be derived as

\[
 u(x, y, t) = \frac{\tan\left(-\frac{\rho}{2(\rho^2 + \sigma^2)}x - \frac{9\rho^5}{2(\rho^2 + \sigma^2)^3}y - \frac{\rho^3}{2(\rho^2 + \sigma^2)^3}\right)}{\rho + \sigma \tan\left(-\frac{\rho}{2(\rho^2 + \sigma^2)}x - \frac{9\rho^5}{2(\rho^2 + \sigma^2)^3}y - \frac{\rho^3}{2(\rho^2 + \sigma^2)^3}\right)}.
\]

It is easy to show that

\[
 u(x, y, t) = \frac{\cot\left(\frac{\rho}{2(\rho^2 + \sigma^2)}x + \frac{9\rho^5}{2(\rho^2 + \sigma^2)^3}y + \frac{\rho^3}{2(\rho^2 + \sigma^2)^3}\right)}{\rho + \sigma \cot\left(\frac{\rho}{2(\rho^2 + \sigma^2)}x + \frac{9\rho^5}{2(\rho^2 + \sigma^2)^3}y + \frac{\rho^3}{2(\rho^2 + \sigma^2)^3}\right)}
\]

is another rational solution of equation (1).

**Rational cos-sin method**

The rational cos-sin approach considers the solution of the B-type KP equation as

\[
 u(x, y, t) = \frac{\rho \cos(kx + ry - \omega t)}{\sigma \cos(kx + ry - \omega t) + \psi \sin(kx + ry - \omega t)}.
\]

By inserting Eq. (8) into Eq. (1) and using a number of operations, we get

\[
 \rho = \frac{2\kappa(\sigma^2 + \psi^2)}{\psi}, \quad \tau = 144\kappa^5, \quad \omega = -4\kappa^3.
\]

Now, the following rational solution to the equation (1) can be extracted

\[
 u(x, y, t) = \frac{2\kappa(\sigma^2 + \psi^2) \cos(kx + 144k^5y + 4\kappa^3t)}{\psi(\sigma \cos(kx + 144k^5y + 4\kappa^3t) + \psi \sin(kx + 144k^5y + 4\kappa^3t))}.
\]
Multiple exp-function method

It is easy to show that through the use of the multiple exp-function method, the following 1, 2, and 3-wave solutions to the B-type KP equation can be constructed

\[ u(x, y, t) = \frac{k_1 e^{k_1x + 9k_2^2y} e^{-k_2^2t}}{1 + e^{k_1x + 9k_2^2y} e^{-k_2^2t}}, \]

\[ u(x, y, t) = \frac{k_1 e^{k_1x + 9k_2^2y} e^{-k_2^2t} + k_2 e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{12}(k_1 + k_2) e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_2x + 9k_2^2y} e^{-k_2^2t}}{1 + e^{k_1x + 9k_2^2y} e^{-k_2^2t} + e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{12} e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_2x + 9k_2^2y} e^{-k_2^2t}}, \]

and

\[ u(x, y, t) = \frac{2(k_1 e^{k_1x + 9k_2^2y} e^{-k_2^2t} + k_2 e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{12}(k_1 + k_2) e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{13}(k_1 + k_3) e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_3x + 9k_2^2y} e^{-k_2^2t} + a_{23}(k_2 + k_3) e^{k_2x + 9k_2^2y} e^{-k_2^2t} e^{k_3x + 9k_2^2y} e^{-k_2^2t} + a_{12} a_{13} a_{23}}{(1 + e^{k_1x + 9k_2^2y} e^{-k_2^2t} + e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{12} e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_2x + 9k_2^2y} e^{-k_2^2t} + a_{13} e^{k_1x + 9k_2^2y} e^{-k_2^2t} e^{k_3x + 9k_2^2y} e^{-k_2^2t} + a_{23} e^{k_2x + 9k_2^2y} e^{-k_2^2t} e^{k_3x + 9k_2^2y} e^{-k_2^2t})}. \]

which

\[ a_{ij} = \frac{(k_i - k_j)^2}{(k_i + k_j)^2}, \quad 1 \leq i, j \leq 3. \]

Figures 3, 4, and 5 show respectively 1, 2 and 3-wave solutions on the x-y plane when \( k_1 = 1, k_2 = -2, k_3 = 3, \) and \( t = 0. \)

Conclusion

In summary, a B-type Kadomtsev–Petviashvili equation presented as a nonlinear model in the fluids was investigated in this work. A group of exact solutions including the N-wave and other solutions to the B-type KP equation was formally extracted by considering the B-type KP equation, its bilinear expression, and exerting capable techniques. Figures in two and three dimensions were provided to demonstrate the characteristics of the solutions. The current research certainly helped to complete the previous studies about the B-type KP equation.
Figure 3. The 1-wave solution on the x-y plane when $k_1 = 1$ and $t = 0$.

Figure 4. The 2-wave solution on the x-y plane when $k_1 = 1$, $k_2 = -2$, and $t = 0$.

Figure 5. The 3-wave solution on the x-y plane when $k_1 = 1$, $k_2 = -2$, $k_3 = 3$, and $t = 0$. 
References

32. Cao, X., Lump solutions to the (3+1)-dimensional generalized b-type Kadomtsev–Petviashvili equation, Advances in Mathematical Physics, 2018 (2018), 7843498
33. Abudiab, M., Khalique, C.M., Exact solutions and conservation laws of a (3+1)-dimensional B-type Kadomtsev–Petviashvili equation, Advances in Difference Equations, 2013 (2013), 221
38. Sedeeg, A.K.H., et al., Generalized optical soliton solutions to the (3+1)-dimensional resonant nonlinear Schrödinger equation with Kerr and parabolic law nonlinearities, Optical and Quantum Electronics, 51 (2019), 173
40. Yépez Martínez, H., Gómez-Aguilar, J.F., Local M-derivative of order \( \alpha \) and the modified expansion function method applied to the longitudinal wave equation in a magneto electro-elastic circular rod, Optical and Quantum Electronics, 50 (2018), 375


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