OSCILLATION PROPERTIES OF SOLUTIONS OF FRACTIONAL DIFFERENCE EQUATIONS

Mustafa BAYRAM[*1] and Aydin SECER[2]

[*1]Department of Computer Engineering, Istanbul Gelisim University

[2]Department of Mathematical Engineering, Yildiz Technical University

[*] Corresponding author; E-mail: mbayram@gelisim.edu.tr

In this article, studied the properties of the oscillation of fractional difference equations, and we obtain some results. The results we obtained are an expansion and further development of highly known results. Then we showed them with examples.

Key words: *Fractional difference equation, oscillatory solutions, oscillation theory.*

1. Introduction and Preliminaries

In the investigations of qualitative properties for differential equations, research on time scales of the dynamic equations, oscillation of differential (or difference) equations and fractional differential equations have been a very important issue in the science and engineering. We refer to [1-25] and the references therein.

We first investigated following fractional difference equations,

$$\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right)+\sum_{i=1}^{n}q_i(t)\left[\sum_{s=t_0}^{t+\alpha-1}\frac{1}{(t-s-1)^{\alpha}}x(s)\right]^{\eta_i}=0. \quad (1)$$

We can rewrite equation (1) as

$$\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right)+\sum_{i=1}^{n}q_i(t)G^{\eta_i}(t)=0. \quad (2)$$

where $t\in N_{t_0+1-\alpha}$, $G(t)=\sum_{s=t_0}^{\alpha+t-1}\frac{1}{(t-s-1)^{\alpha}}x(s)$, δ_1, δ_2 and η_i are the division of two odd positive integers. $\psi(t)$, $a(t)$ and $q_i(t)$ are positive coefficient sequences and Δ^{α} demonstrate that the Riemann-Liouville fractional difference operator of order α where $0<\alpha\le 1$. Therefore, in our results we use the following conditions:

C1. $\sum_{s=t_0}^{\infty}\left(1/\psi^{1/\delta_1}(s)\right)=\infty$ and $\sum_{s=t_0}^{\infty}\left(1/a^{1/\delta_2}(s)\right)=\infty$, (3)

C2. $\sum_{s=t_0}^{\infty}\left(1/\psi^{1/\delta_1}(s)\right)<\infty$ and $\sum_{s=t_0}^{\infty}\left(1/a^{1/\delta_2}(s)\right)<\infty$. (4)

By a solution of equation (2), we mean a real-valued sequence $x(t)$ satisfying equation (2) for

$t \in \mathbb{N}_{t_0}$. A solution $x(t)$ of equation (2) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called non-oscillatory. Equation (2) is called oscillatory if all its solutions are oscillaory.

Definition 1. [26]. We define *v*th fractional sum f as

$$\Delta^{-v} f(t) = \left[\Gamma(v)\right]^{-1} \sum_{s=a}^{t-v} (t-s-1)^{v-1} f(s), \quad v > 0$$

(5)

where we define f for $s \equiv a \bmod(2)$, $\Delta^{-v} f$ for $t \equiv (a+v) \bmod(2)$ and $t^{(v)} = \frac{\Gamma(1+t)}{\Gamma(1+t-v)}$. The fractional sum $\Delta^{-v} f$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+v}$, where $\mathbb{N}_t = \{t,\ t+1,\ t+2,...\}$.

Definition 2. [26] Let $m-1 < \mu < m$ and $v > 0$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $v = m - \mu$. Then we define that μ-th fractional difference as

$$\Delta^{\mu} f(t) = \Delta^{m-v} f(t) = \Delta^{m} \Delta^{-v} f(t). \tag{6}$$

2. Oscillation Properties of (2)

In this section, we work the oscillation properties of (2).

Lemma 1. [22]. Suppose that $x(t)$ be a solution of equation (2) and let

$$G(t) = \sum_{s=t_0}^{\alpha+t-1} (t-s-1)^{(-\alpha)} x(s) \tag{7}$$

then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha} x(t). \tag{8}$$

Theorem 1. Assume C1 holds and furthermore, for all suficiently large t,

$$\sum_{s=t_3}^{\infty} \left(\frac{1}{\psi(s)} \sum_{\tau=s}^{\infty} \left(\sum_{i=1}^{n} \frac{1}{a(\tau)} \sum_{\zeta=\tau}^{\infty} q_i(\zeta) \right)^{1/\delta_2} \right)^{1/\delta_1} = \infty \tag{9}$$

and

$$\sum_{i=1}^{n} \sum_{s=t_3}^{\infty} q_i(s) \left(\sum_{\tau=t_2}^{s-1} \frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)} \right)^{\eta_i} = \infty \tag{10}$$

Then every solution of (2) is either oscillatory or $\lim_{t\to\infty} G(t) = 0$.

Proof. Assume that the contrary that $x(t)$ is non-oscillatory solution of (2). Then without loss of generality, we may assume that there is a solution $x(t)$ of (2) such that $x(t) > 0$ on $[t_1, \infty)$, where t_1 is sufficiently large, so that $G(t) > 0$ on $[t_1, \infty)$. And all of $q_i(t)$'s are not identically zero on $[t_1, \infty)$ for $i = 1, 2, ..., n$. From (2), we have

$$\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right)=-\sum_{i=1}^{n}q_i(t)G^{\eta_i}(t)<0. \qquad (11)$$

In that case $a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}$ is an eventually non-increasing sequence on $[t_1,\infty)$. So, we understand that $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)$ and $\Delta^{\alpha}x(t)$ are ultimately of one sign. For $t_2>t_1$ is big enough, $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}\right)$ and $\Delta^{\alpha}x(t)$ have a fixed sign on $[t_2,\infty)$. We then consider the following conditions:

Case 1. $\Delta^{\alpha}x(t)<0$ and $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)<0$;

Case 2. $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)<0$ and $0<\Delta^{\alpha}x(t)$;

Case 3. $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)>0$ and $0>\Delta^{\alpha}x(t)$;

Case 4. $\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)>0$ and $0<\Delta^{\alpha}x(t)$.

For the Case 1, we have

$$\frac{G(t)}{\Gamma(1-\alpha)}=\frac{G(t_2)}{\Gamma(1-\alpha)}+\sum_{s=t_2}^{t-1}\frac{\left(\psi(s)\left(\Delta^{\alpha}x(s)\right)^{\delta_1}\right)^{1/\delta_1}}{\psi^{1/\delta_1}(s)}\leq\frac{G(t_2)}{\Gamma(1-\alpha)}+\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}.$$

Then, by C1, we obtain $\lim_{t\to\infty}G(t)=-\infty$ which contradicts with $0<G(t)$.

For the Case 2, we have from (9),

$$\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}=\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}+\sum_{s=t_2}^{t-1}\frac{\left(a(s)\left[\Delta\left(\psi(s)\left(\Delta^{\alpha}x(s)\right)^{\delta_1}\right)\right]^{\delta_2}\right)^{1/\delta_2}}{a^{1/\delta_2}(s)}$$

$$\leq\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}+\left(a(t_2)\left[\Delta\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)\right]^{\delta_2}\right)^{1/\delta_2}\sum_{s=t_2}^{t-1}\frac{1}{a^{1/\delta_2}(s)}.$$

Then, by C1, we obtain $\lim_{t\to\infty}\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\gamma_1}=-\infty$ which contradicts with $0<\Delta^{\alpha}x(t)$.

For the Case 3, we have $\lim_{t\to\infty}G(t)=k_1\geq 0$ and $\lim_{t\to\infty}\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}=k_2\leq 0$. If we suppose that $k_1>0$, then $G(t)>k_1$ for $t\leq t_3\leq t_2$. Therefore, If we sum both sides of (2) from t to ∞, we obtain

$$a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}<-\sum_{s=t}^{\infty}\sum_{i=1}^{n}q_i(s)G^{\eta_i}(s)\leq-\sum_{i=1}^{n}k_1^{\eta_i}\sum_{s=t}^{\infty}q_i(s)$$

that is

$$\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\leq-\left(\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(t)}\sum_{s=t}^{\infty}q_i(s)\right)^{1/\delta_2}. \qquad (12)$$

If we sum both sides of the (12) from t to ∞, we have

$$\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}-k_2\le-\sum_{s=t}^{\infty}\left(\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(s)}\sum_{\tau=s}^{\infty}q_i(\tau)\right)^{1/\delta_2}$$

which means for $k_2\le 0$

$$\Delta^{\alpha}x(t)\le-\left(\left(\psi(t)^{-1}\right)\sum_{s=t}^{\infty}\left(\sum_{i=1}^{n}k_1^{\eta_i}\left(a(s)^{-1}\right)\sum_{\tau=s}^{\infty}q_i(\tau)\right)^{1/\delta_2}\right)^{1/\delta_1}. \tag{13}$$

If we sum both sides of the (13) from t_3 to $t-1$, we obtain

$$\frac{G(t)}{\Gamma(1-\alpha)}\le\frac{G(t_3)}{\Gamma(1-\alpha)}-\sum_{s=t_3}^{t-1}\left(\frac{1}{\psi(s)}\sum_{\tau=s}^{\infty}\left(\sum_{i=1}^{n}\frac{k_1^{\eta_i}}{a(\tau)}\sum_{\zeta=\tau}^{\infty}q_i(\zeta)\right)^{1/\delta_2}\right)^{1/\delta_1}.$$

Therefore, by (9), we obtain $\lim_{t\to\infty}G(t)=-\infty$ with contradicts with $G(t)>0$.

For the Case 4, we have

$$\frac{G(t)}{\Gamma(1-\alpha)}=\frac{G(t_2)}{\Gamma(1-\alpha)}+\sum_{s=t_2}^{t-1}\frac{\left(\psi(s)\left(\Delta^{\alpha}x(s)\right)^{\delta_1}\right)^{1/\delta_1}}{\psi^{1/\delta_1}(s)}>\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)^{1/\gamma_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}>0.$$

That is,

$$\left(\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(s)}\right)^{\eta_i}<G^{\eta_i}(t).$$

Then from (2),

$$\sum_{i=1}^{n}q_i(t)\left(\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(s)}\right)^{\eta_i}<-\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right). \tag{14}$$

If we sum both sides of the (14) from t_3 to $t-1$, we obtain

$$\sum_{i=1}^{n}\left(\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)^{1/\delta_1}\right)^{\eta_i}\sum_{s=t_3}^{t-1}q_i(s)\left(\sum_{\tau=t_2}^{s-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)}\right)^{\eta_i}<a(t_3)\left[\Delta\left(\psi(t_3)\left(\Delta^{\alpha}x(t_3)\right)^{\delta_1}\right)\right]^{\delta_2}.$$

If we take $t\to\infty$, we get a contradiction with equation (10). Therefore, the proof of the Theorem 1 is complete

Theorem 2. Suppose that C2, (9) and (10) hold. Furthermore, for all sufficiently large t,

$$\sum_{s=t_4}^{\infty}\left(\frac{\Gamma(1-\alpha)}{\psi(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}\frac{1}{a(\tau)}\sum_{\zeta=t_2}^{\tau-1}q_i(\zeta)\left[\sum_{\xi=\zeta}^{\infty}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\xi)}\right]^{\eta_i}\right)^{1/\delta_2}\right)^{1/\delta_1}=\infty \tag{15}$$

and

$$\sum_{s=t_4}^{\infty}\left(\frac{1}{a(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}q_i(\tau)\left(\left(\sum_{\zeta=\tau}^{\infty}\frac{1}{a^{1/\delta_2}(\zeta)}\right)^{1/\delta_1}\sum_{\zeta=t_2}^{\tau-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\zeta)}\right)^{\eta_i}\right)\right)^{1/\delta_2}=\infty \tag{16}$$

Therefore, each solution of (2) is either $\lim_{t\to\infty}G(t)=0$ or oscillatory.

Proof. Let's the contrary that $x(t)$ is non-oscillatory solution of (2). Then without loss of generality, we assume that there is a solution $x(t)$ of (2) such that $0<x(t)$ on $[t_1,\infty)$, where t_1 is sufficiently large, so that $G(t)>0$ on $[t_1,\infty)$. It appears that all of $q_i(t)$'s are not identically zero on $[t_1,\infty)$ for $i=1,2,...,n$. From (11), we obtained that $a(t)\left[\Delta\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)\right]^{\delta_2}$ is an eventually non-increasing sequence on $[t_1,\infty)$. For the Case 1, we have

$$-\frac{G(t)}{\Gamma(1-\alpha)}<\sum_{s=t}^{\infty}\left(\left(\psi(s)\left(\Delta^\alpha x(s)\right)^{\delta_1}\right)^{1/\delta_1}\left(\psi^{1/\delta_1}(s)\right)^{-1}\right)<\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t}^{\infty}\left(\psi^{1/\delta_1}(s)\right)^{-1}$$

$$<\left(\psi(t_1)\left(\Delta^\alpha x(t_1)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t}^{\infty}\left(\psi^{1/\delta_1}(s)\right)^{-1}=K_1\sum_{s=t}^{\infty}\left(\psi^{1/\delta_1}(s)\right)^{-1}.$$

Then from the last inequality and (2), we obtain

$$\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right)<\sum_{i=1}^{n}q_i(t)\left[\Gamma(1-\alpha)K_1\sum_{s=t}^{\infty}\frac{1}{\psi^{1/\delta_1}(s)}\right]^{\eta_i}. \tag{17}$$

If we sum both sides of the (17) from t_2 to $t-1$,

$$a(t)\left[\Delta\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)\right]^{\delta_2}<\sum_{i=1}^{n}\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}\sum_{s=t_2}^{t-1}q_i(s)\left[\sum_{\tau=s}^{\infty}\frac{1}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i},$$

that is

$$\Delta\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)<\left(\sum_{i=1}^{n}\frac{\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}}{a(t)}\sum_{s=t_3}^{t-1}q_i(s)\left[\sum_{\tau=s}^{\infty}\frac{1}{\psi^{1/\delta_1}(\tau)}\right]^{\eta_i}\right)^{1/\delta_2}. \tag{18}$$

If we sum both sides of the (18) from t_3 to $t-1$,

$$\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}<\sum_{s=t_3}^{t-1}\left(\sum_{i=1}^{n}\frac{\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}}{a(s)}\sum_{\tau=t_2}^{s-1}q_i(\tau)\left[\sum_{\zeta=\tau}^{\infty}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right]^{\eta_i}\right)^{1/\delta_2}$$

then we get

$$\Delta G(t)<\left(\frac{\Gamma(1-\alpha)}{\psi(t)}\sum_{s=t_3}^{t-1}\left(\sum_{i=1}^{n}\frac{\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}}{a(s)}\sum_{\tau=t_2}^{s-1}q_i(\tau)\left[\sum_{\zeta=\tau}^{\infty}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right]^{\eta_i}\right)^{1/\delta_2}\right)^{1/\delta_1}. \tag{19}$$

If we sum both sides of the the (19) from t_4 to $t-1$, we have

$$G(t)-G(t_4)<\sum_{s=t_4}^{t-1}\left(\frac{\Gamma(1-\alpha)}{\psi(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}\frac{\left(\Gamma(1-\alpha)K_1\right)^{\eta_i}}{a(\tau)}\sum_{\zeta=t_2}^{\tau-1}q_i(\zeta)\left[\sum_{\xi=\zeta}^{\infty}\frac{1}{\psi^{1/\delta_1}(\xi)}\right]^{\eta_i}\right)^{1/\delta_2}\right)^{1/\delta_1}.$$

By (14), we obtain $\lim_{t\to\infty}G(t)=-\infty$ due to $K_1<0$, which conradicts with $0<G(t)$.

For the Case 2,

$$G(t)>\sum_{s=t_2}^{t-1}\frac{\Gamma(1-\alpha)\left(\psi(s)\left(\Delta^\alpha x(s)\right)^{\delta_1}\right)^{1/\delta_1}}{\psi^{1/\delta_1}(s)}>\Gamma(1-\alpha)\left(\psi(t)\left(\Delta^\alpha x(t)\right)^{\delta_1}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}$$

and

$$-\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1} \le \sum_{s=t}^{\infty}\frac{\left(a(s)\left(\Delta\left(\psi(s)\left(\Delta^{\alpha}x(s)\right)^{\delta_1}\right)\right)^{\delta_2}\right)^{1/\delta_2}}{a^{1/\delta_2}(s)}$$

$$< \left(a(t)\left(\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right)^{\delta_2}\right)^{1/\delta_2}\sum_{s=t}^{\infty}\frac{1}{a^{1/\delta_2}(s)}$$

$$< \left(a(t_2)\left(\Delta\left(\psi(t_2)\left(\Delta^{\alpha}x(t_2)\right)^{\delta_1}\right)\right)^{\delta_2}\right)^{1/\delta_2}\sum_{s=t}^{\infty}\frac{1}{a^{1/\delta_2}(s)}$$

$$= K_2\sum_{s=t}^{\infty}\frac{1}{a^{1/\delta_2}(s)}.$$

Thereore, we have

$$G(t) > -\Gamma(1-\alpha)\left(K_2\sum_{s=t}^{\infty}\frac{1}{a^{1/\delta_2}(s)}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}.$$

Thus, from (2), we obtain

$$\Delta\left(a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2}\right) = \sum_{i=1}^{n}q_i(t)\left(\Gamma(1-\alpha)\left(K_2\sum_{s=t}^{\infty}\frac{1}{a^{1/\delta_2}(s)}\right)^{1/\delta_1}\sum_{s=t_2}^{t-1}\frac{1}{\psi^{1/\delta_1}(s)}\right)^{\eta_i}. \quad (20)$$

If we sum two sides of the (20) from t_3 to $t-1$, we have

$$a(t)\left[\Delta\left(\psi(t)\left(\Delta^{\alpha}x(t)\right)^{\delta_1}\right)\right]^{\delta_2} = \sum_{s=t_3}^{t-1}\left(\sum_{i=1}^{n}q_i(s)\left(\Gamma(1-\alpha)\left(K_2\sum_{\tau=s}^{\infty}\frac{1}{a^{1/\delta_2}(\tau)}\right)^{1/\delta_1}\sum_{\tau=t_2}^{s-1}\frac{1}{\psi^{1/\delta_1}(\tau)}\right)^{\eta_i}\right).$$

Then

$$\psi(t_4)\left(\Delta^{\alpha}x(t_4)\right)^{\delta_1} = \sum_{s=t_4}^{t-1}\left(\frac{1}{a(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}q_i(\tau)\left(\Gamma(1-\alpha)\left(K_2\sum_{\zeta=\tau}^{\infty}\frac{1}{a^{1/\delta_2}(\zeta)}\right)^{1/\delta_1}\sum_{\zeta=t_2}^{\tau-1}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right)^{\eta_i}\right)\right)^{1/\delta_2}$$

letting $t\to\infty$, we obtain

$$\sum_{s=t_4}^{\infty}\left(\frac{1}{a(s)}\sum_{\tau=t_3}^{s-1}\left(\sum_{i=1}^{n}q_i(\tau)\left(\Gamma(1-\alpha)\left(K_2\sum_{\zeta=\tau}^{\infty}\frac{1}{a^{1/\delta_2}(\zeta)}\right)^{1/\delta_1}\sum_{\zeta=t_2}^{\tau-1}\frac{1}{\psi^{1/\delta_1}(\zeta)}\right)^{\eta_i}\right)\right)^{1/\delta_2} < \infty$$

which contradicts with (16). The rest of the proof is made similar to the proof of the Theorem 1. Thus the proof of the theorem is completed.

3. Application

Let's consider the following fractional difference equation as an example

$$\Delta\left(t^{1/2}\sqrt{\left[\Delta\left(\left(\Delta^{\alpha}x(t)\right)^{7}\right)\right]}\right) + \frac{1}{t^2}\left(\sum_{s=t_0}^{t-1+\alpha}\left[-(s+1-t)\right]^{(-\alpha)}x(s)\right)^{3} = 0,\ 2\le t. \quad (21)$$

This corresponds to (2) with $t_0 = 2$, $\delta_1 = 7$, $\delta_2 = 1/2$, $\alpha \in (0,1]$, $a(t) = t^{1/2}$, $\psi(t) = 1$, $q(t) = 1/t^2$, $n = 1$, and $\eta_1 = 3$. However,

$$\sum_{s=t_0}^{\infty}\frac{1}{\psi^{1/\gamma_1}(s)}=\sum_{s=t_0}^{\infty}1=\infty$$

and

$$\sum_{s=t_0}^{\infty}\frac{1}{a^{1/\delta_2}(s)}=\sum_{s=t_2}^{\infty}\frac{1}{s}=\infty.$$

Then C1 holds. So, we have

$$\sum_{s=t_3}^{\infty}\left(\frac{1}{\psi(s)}\sum_{\tau=s}^{\infty}\left(\sum_{i=1}^{n}\frac{1}{a(\tau)}\sum_{\zeta=\tau}^{\infty}q_i(\zeta)\right)^{1/\delta_2}\right)^{1/\delta_1}=\sum_{s=t_3}^{\infty}\left(\sum_{\tau=s}^{\infty}\left(\frac{1}{\tau^{1/2}}\sum_{\zeta=\tau}^{\infty}\zeta^{-2}\right)^{2}\right)^{1/7}=\infty$$

and

$$\sum_{i=1}^{n}\sum_{s=t_3}^{\infty}q_i(s)\left(\sum_{\tau=t_2}^{s-1}\frac{\Gamma(1-\alpha)}{\psi^{1/\delta_1}(\tau)}\right)^{\eta_i}=\sum_{s=t_3}^{\infty}\frac{1}{s^2}\left(\sum_{\tau=2}^{s-1}\Gamma(1-\alpha)\right)^{3}=\sum_{s=t_3}^{\infty}\frac{1}{s^2}\left(\Gamma(1-\alpha)(s-2)\right)^{3}=\infty.$$

Therefore, (9) and (10) holds, and then we say that (21) is $\lim_{t\to\infty}G(t)=0$ or oscillatory by Theorem 1.

Counclusion

In this work, we studied the qualitative behavior of solutions of nonlinear fractional difference equations (FDE) with fractional Riemann–Liouville difference operator. Because there was a gap for the oscillatory solutions of FDE under the condition (C2) in the literature, we considered the equation with the conditions (C1) and (C2). By using some techniques, we obtained some oscillation results. The obtained results improved the many criteria in the literatüre.

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