We introduce double $\lambda$-statistically convergent sequences and double $\lambda$-statistically Cauchy sequences in the fuzzy normed spaces. We study $[V,\bar{\lambda}]$ and $[C,1]$-summabilities for double sequences. In addition, we obtain the relation between these concepts and $\bar{\lambda}$-statistically convergence.

Key words: $\bar{\lambda}$-convergence; double sequence; summabilities

1. Introduction

Theory of statistical convergence was firstly originated by Fast [1]. After Fridy [2] and Šalát [3], statistical convergence became a notable topic in summability theory.

Fuzzy set theory has become an important working area for 40 years. It has been used in many engineering applications, control of chaos, non-linear operator, population changes. It affected many mathematicians to investigate new kinds of sequence spaces and to study this type convergence. Fuzzy norm idea was firstly considered by Katsaras [4] in the fuzzy topological vector spaces. Also Alimohammady and Roohi [5] introduced compactness in fuzzy minimal spaces. Felbin [6] was inspired by Kaleva and Seikkala [7], and then he introduced fuzzy norm of the linear space. Topological characterizations of fuzzy normed linear spaces were found in [8,9]. Other studies on the same spaces can also be found in [10,11].

The convergence of a sequence using fuzzy numbers was given by Matloka [12]. Nanda [13] studied the sequences with fuzzy numbers. He also formed complete metric space using set of all convergent sequences with fuzzy numbers. Mursaleen and Edely [14] introduced statistical convergence via double sequences and identified some results about statistical convergence. Çakan and Altay [15], Altay and Başar [16], Tripathy [17] studied double sequences in summability theory. Statistical convergence using fuzzy numbers was given by Nuray and Savaş [18]. Savaş and Mursaleen [19] studied statistical convergent double sequences via fuzzy numbers. Şençimen and Pehlivan [20] introduced statistically convergent (resp. statistically Cauchy) sequence in the fuzzy normed linear space. Considering [20], Mohiuddine et al. [21] studied double sequences in the same spaces.

Mursaleen [22] introduced the idea of $\lambda$-statistical convergence. Using fuzzy numbers, $\lambda$-statistical convergence was presented Savaş [23]. $\lambda$-statistical convergence by using double sequences was obtained by Savaş [24], and Savaş and Patterson [25,26]. Türkmen and Çınar [27] considered $\lambda$-statistical convergence within fuzzy normed linear space.
Main results

We define double $\bar{\lambda}$-statistically convergence sequence, double $\bar{\lambda}$-statistically Cauchy sequence, $[V, \bar{\lambda}]$ and $[C.1]$-summabilities for double sequences in the fuzzy normed spaces. Also, we give notable properties and relationships between these concepts.

Throughout the paper, we consider $(X, \| \|)$ be and $FNS$ and $\bar{\lambda}_{r,s} = \lambda_r \mu_s$ be the collection of such sequences $\bar{\lambda}$ will be represented by $A_2$.

Let $\lambda = (\lambda_r)$ and $\mu = (\mu_s)$ be non-decreasing sequences of $\mathbb{R}^+$, each converges to $\infty$ and such that $\lambda_{r+1} \leq \lambda_r + 1, \lambda_1 = 1; \mu_{s+1} \leq \mu_s + 1, \mu_1 = 1$. Let $I_r = [r - \lambda_r + 1, r]$, $I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$.

For any set $X \subset \mathbb{N} \times \mathbb{N}$, following number

$$\delta_{\bar{\lambda}}(X) = \left| P - \lim_{r,s \to \infty} \frac{1}{I_{r,s}} \left| \{(k, l) \in I_{r,s}: (k, l) \in X \} \right| \right|$$

is called $\bar{\lambda}$-density of $X$, where the limit exists and then $\bar{\lambda}_{r,s} = \lambda_r \mu_s$.

Next, we give double $\bar{\lambda}$-statistically convergence in fuzzy normed space.

Definition 1. $x = (x_{k,l})$ in $X$ is called double $\bar{\lambda}$-statistically convergent to $L \in X$ with regards to fuzzy norm on $X$ or $FS_{\bar{\lambda}}$-convergent if for each $\varepsilon > 0$

$$\lim_{r,s \to \infty} \frac{1}{I_{r,s}} \left| \left\{ (k, l) \in I_{r,s} : \| x_{k,l} - L \|_0^+ \geq \varepsilon \right\} \right| = 0.$$

It is demonstrated by $x_{k,l} \xrightarrow{FS_{\bar{\lambda}}} L$ or $x_{k,l} \xrightarrow{L} (FS_{\bar{\lambda}})$ or $FS_{\bar{\lambda}} - \lim_{k,l \to \infty} x_{k,l} = L$ where $I_r = [r - \lambda_r + 1, r]$, $I_s = [s - \mu_s + 1, s]$ and $I_{r,s} = I_r \times I_s$.

For each $\varepsilon > 0$,

$$K(\varepsilon) = \left\{ (k, l) \in I_{r,s} : \| x_{k,l} - L \|_0^+ \geq \varepsilon \right\}$$

has natural zero density. That is, $\| x_{k,l} - L \|_0^+ < \varepsilon$ for almost all $k, l$. It is denoted by

$$FS_{\bar{\lambda}} - \lim_{k,l \to \infty} x_{k,l} = L.$$

In $X$, all $\bar{\lambda}$-statistically convergent sequences is described by $FS_{\bar{\lambda}}(X)$, and also defined by as follows:

$$FS_{\bar{\lambda}}(X) = \left\{ x = (x_{k,l}) : \text{for some} \ L, FS_{\bar{\lambda}} - \lim_{k,l \to \infty} x_{k,l} = L \right\}.$$

$L \in X$ is $FS_{\bar{\lambda}}$-limit of a $(x_{k,l})$.

In terms of neighborhoods, we have $x_{k,l} \xrightarrow{FS_{\bar{\lambda}}} L$ if for each $\varepsilon > 0$,

$$\delta_{\bar{\lambda}}(\{(k, l) \in I_{r,s} : x_{k,l} \notin N_L(\varepsilon, 0)\}) = 0,$$

that is, for each $\varepsilon > 0, x_{k,l} \notin N_L(\varepsilon, 0)$ for almost all $(k, l)$.

A useful result of the above definition is:

$$x_{k,l} \xrightarrow{FS_{\bar{\lambda}}} L \text{ iff } FS_{\bar{\lambda}} - \lim_{k,l \to \infty} \| x_{k,l} - L \|_0^+ = 0.$$

Note that $FS_{\bar{\lambda}} - \lim_{k,l \to \infty} \| x_{k,l} - L \|_0^+ = 0$ implies that

$$FS_{\bar{\lambda}} - \lim_{k,l \to \infty} \| x_{k,l} - L \|_\alpha = FS_{\bar{\lambda}} - \lim_{k,l \to \infty} \| x_{k,l} - L \|_\alpha^+ = 0,$$
for each $\alpha \in [0,1]$ since

$$0 \leq \|x_{k,l} - L\|_\alpha \leq \|x_{k,l} - L\|_\alpha^+ \leq \|x_{k,l} - L\|_0^+$$

holds for every $(k, l) \in I_{r,s}$ and for each $\alpha \in [0,1]$.

For all $r,s$, if $\overline{\lambda}_{r,s} = r,s$, set of $\overline{\lambda}$-statistically convergent obtained by double sequences transforms to set of statistically convergent via double sequences in $X$.

**Definition 2.** $x = (x_{k,l})$ in $X$ is called strongly double $\overline{\lambda}$-summable with regards to fuzzy norm on $X$ if there is a $L \in X$ such that

$$\lim_{r,s \to \infty} \frac{1}{r,s} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, 0) = 0.$$ 

It is denoted by $x_{k,l} \xrightarrow{[V, \overline{\lambda}]_{FN}} L$ or $x_{k,l} \to L \left( [V, \overline{\lambda}]_{FN} \right)$ or $[V, \overline{\lambda}]_{FN} - \lim_{k,l \to \infty} x_{k,l} = L$.

If $\overline{\lambda}_{r,s} = r,s$, strongly double $\overline{\lambda}$-summable tranforms to $[C, 1]_{FN}$, the space of strongly Cesàro summable with double sequences in fuzzy normed space. It is defined by as follows:

$$\lim_{r,s \to \infty} \sum_{k,l = 1,1}^{r,s} (D\|x_{k,l} - L\|, 0) = 0.$$ 

So, we have written

$$[V, \overline{\lambda}]_{FN}(X) = \left\{ x = (x_{k,l}) : \lim_{r,s \to \infty} \frac{1}{r,s} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, 0) = 0, \text{ for some } L \right\}$$

$$[C, 1]_{FN}(X) = \left\{ x = (x_{k,l}) : \lim_{r,s \to \infty} \frac{1}{r,s} \sum_{k,l \in I_{r,s}}^{r,s} (D\|x_{k,l} - L\|, 0) = 0, \text{ for some } L \right\}$$

**Definition 3.** $x = (x_{k,l})$ in $X$ is called $FS_{\overline{\lambda}}$-convergent to $L \in X$ with regards to fuzzy norm on $X$ if for each $\varepsilon > 0$ and $t \in (0,1)$,

$$\delta_{\overline{\lambda}} = \left\{ \{(k,l) \in I_{r,s} : D(\|x_{k,l} - L\|, 0) \leq 1 - t) \right\} = 0$$

or equivalently

$$\delta_{\overline{\lambda}} = \left\{ \{(k,l) \in I_{r,s} : D(\|x_{k,l} - L\|, 0) > 1 - t) \right\} = 1$$

**Theorem 1.** If $FS_{\overline{\lambda}} \sim x_{k,l} = L$ exists, it is unique.

Proof. Let $L_1, L_2 (L_1 \neq L_2)$ be in $X$ such that

$$FS_{\overline{\lambda}} \sim x_{k,l} = L_1; FS_{\overline{\lambda}} \sim x_{k,l} = L_2.$$ 

If $L_1 \neq L_2$, then $L_1 - L_2 \neq 0$. Therefore, $\|L_1 - L_2\|_0^+ = 2\varepsilon > 0$, and we take a norm as $\|\|_0^+$.

Since $FS_{\overline{\lambda}} \sim x_{k,l} = L_1$ and $FS_{\overline{\lambda}} \sim x_{k,l} = L_2$ it follows that

$$\lim_{r,s \to \infty} \frac{1}{r,s} \left\{ \{(k,l) \in I_{r,s} : \|x_{k,l} - L_1\|_0^+ \geq \varepsilon) \right\} = 0$$

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There are \((k, l) \in I_{r,s}\) such that 
\[
\|x_{k,l} - L_1\|_0^+ + \|x_{k,l} - L_2\|_0^+ \leq 2\varepsilon
\]
which is a contradiction. Therefore, the limit unique.

**Theorem 2.** Let \((X, \|\cdot\|)\) be an FNS, \(\bar{\lambda} \in \Delta_2\). Then

(i) If \(x_{k,l} \to L\left([V, \bar{\lambda}]_{FN}\right)\), then \(x_{k,l} \to L(\text{FS}_X)\) with regards to fuzzy norm on \(X\).

(ii) \([V, \bar{\lambda}]_{FN}(X)\) is proper subset of \(\text{FS}_X(X)\).

Proof. (i) Let \(\varepsilon > 0\) and \(x_{k,l} \to L\left([V, \bar{\lambda}]_{FN}\right)\), we can write

\[
\sum_{k,l \in I_{r,s}} \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right)
\]

\[
\geq \sum_{k,l \in I_{r,s}} \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right) \\
\text{subject to: } \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right) \geq \varepsilon
\]

\[
\geq \varepsilon \cdot \left\{(k, l) \in I_{r,s} : \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right) \geq \varepsilon \right\}
\]

and so

\[
\frac{1}{\varepsilon \overline{\lambda}_{r,s}} \sum_{k,l \in I_{r,s}} \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right)
\]

\[
\geq \frac{1}{\overline{\lambda}_{r,s}} \left\{(k, l) \in I_{r,s} : \left(D\|x_{k,l} - L\|_0^+, \overline{0}\right) \geq \varepsilon \right\}. 
\]

This implies that if \(x_{k,l} \to L\left([V, \bar{\lambda}]_{FN}\right)\), then \(x_{k,l} \to L(\text{FS}_X)\) in fuzzy normed space. This completes the proof.

(ii) To show the inclusion \([V, \bar{\lambda}]_{FN}(X) \subset \text{FS}_X(X)\) is a proper, we define \(x = (x_{k,l})\) by
\[ x_{k,l} = \begin{cases} 
(k,l), & \text{if } k_{r-1} \leq k < k_{r-1} + \sqrt{\lambda_r}, l_{s-1} \leq l < l_{s-1} + \sqrt{\mu_s}, \\
(0,0), & \text{otherwise.} 
\end{cases} 
(r,s = 1,2,\ldots) \]

We see that \( x \) is not bounded. For every \( \varepsilon > 0 \), \( x \in X \), we obtain
\[
\frac{1}{x_{r,s}} \left[ \{ (k,l) \in I_{r,s} : D(\|x_{k,l} - L\|, \bar{0}) \geq \varepsilon \} \right] \leq \frac{\sqrt{\lambda_r} \sqrt{\mu_s}}{x_{r,s}} \to 0, \ (r,s \to \infty).
\]
That is, \( x_{k,l} \to 0 \). On the other side,
\[
\frac{1}{x_{r,s}} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - 0\|, \bar{0})
= \frac{1}{x_{r,s}} \sum_{k,l \in I_{r,s}} \|x_{k,l}\|^+_0
= \frac{1}{x_{r,s}} \cdot \frac{\lfloor \sqrt{\lambda_r} \cdot \lfloor \sqrt{\mu_s} + 1 \rfloor \cdot \lfloor \sqrt{\lambda_r} \rfloor \cdot \lfloor \sqrt{\mu_s} + 1 \rfloor}{4} \to \frac{1}{4} \neq 0.
\]

Hence, \( x_{k,l} \to 0 \left( [V, \bar{\lambda}]_{FN} \right) \).

**Theorem 3.** Let a bounded \( (x_{k,l}) \) is double \( \bar{\lambda} \)-statistically convergent to \( L \). Hence, it is strongly double \( \bar{\lambda} \)-summable to \( L \). Therefore, \( (x_{k,l}) \) is double Cesàro summable to \( L \) with regards to fuzzy norm on \( X \).

Proof. Assume that \( (x_{k,l}) \) is bounded and \( x_{k,l} \to L \left( FS_{\bar{\lambda}} \right) \). Since \( (x_{k,l}) \) is bounded, we get \( D(\|x_{k,l} - L\|, \bar{0}) \leq M \) for all \( k,l \). For \( \varepsilon > 0 \) and for large \( r,s \) we get
\[
\frac{1}{x_{r,s}} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0})
= \frac{1}{x_{r,s}} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0}) + \frac{1}{x_{r,s}} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0})
\leq \frac{M}{x_{r,s}} \left\{ (k,l) \in I_{r,s} : D(\|x_{k,l} - L\|, \bar{0}) \geq \frac{\varepsilon}{2} \right\} + \frac{\varepsilon}{2}.
\]

This implies that \( x_{k,l} \to L \left( [V, \bar{\lambda}]_{FN} \right) \).
Further, we have
\[
\frac{1}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0}) = \frac{1}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0}) + \frac{1}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0})
\]
\[
\leq \frac{1}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0}) + \frac{1}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0})
\]
\[
\leq \frac{2}{rs} \sum_{k,l \in I_{r,s}} (D\|x_{k,l} - L\|, \bar{0}).
\]

Hence, \((x_{k,l})\) is double Cesàro summable to \(L\) since \((x_{k,l})\) strongly double \(\lambda\)-summable to \(L\) in fuzzy normed space.

**Theorem 4.** Let \((x_{k,l})\) be double statistically convergent to \(L\) in fuzzy normed space. Then, it is double \(\lambda\)-statistically convergent to \(L\) with regards to the fuzzy norm on \(X\) iff
\[
P - \lim_{r,s \to \infty} \inf \frac{f_{rs}}{rs} > 0. \tag{1}
\]

Proof. For \(c > 0\), we obtain
\[
\left\{(k,l), k \leq r, l \leq s : (D\|x_{k,l} - L\|, \bar{0}) \geq \varepsilon\right\}
\]
\[
\subseteq \left\{(k,l) \in I_{r,s} : D\left(\|x_{k,l} - L\|, \bar{0}\right) \geq \varepsilon\right\}.
\]

Therefore,
\[
\frac{1}{rs} \left\|(k,l), k \leq r, l \leq s : (D\|x_{k,l} - L\|, \bar{0}) \geq \varepsilon\right\|
\]
\[
\geq \frac{1}{rs} \left\|(k,l) \in I_{r,s} : D\left(\|x_{k,l} - L\|, \bar{0}\right) \geq \varepsilon\right\|
\]
\[
\geq \frac{\lambda_{rs}}{rs} \frac{1}{\lambda_{rs}} \left\|(k,l) \in I_{r,s} : (D\|x_{k,l} - L\|, \bar{0}) \geq \varepsilon\right\|.
\]

Using limit as \(r, s \to \infty\), we get \(x_{k,l} \xrightarrow{FS_s^2} L\). That is, \(x_{k,l} \xrightarrow{st_2(FN)} L \Rightarrow x_{k,l} \xrightarrow{FS_s^2} L\).

Conversely, assume that \(x \in st_2(FN)(X)\) and since \(\lambda_{r,s} = \lambda_r \mu_s\) either \(P - \lim_{r \to \infty} \frac{\lambda_r}{r} = 0\) or \(P - \lim_{s \to \infty} \frac{\lambda_s}{s} = 0\) or both of them are zero. Hence, choosing subsequences \(\left(r(p)\right)_{p=1}^{\infty}\) and \(\left(s(q)\right)_{q=1}^{\infty}\) such that \(\frac{\lambda_{r(p)}}{r(p)} < \frac{1}{p}\) and \(\frac{\lambda_{s(q)}}{s(q)} < \frac{1}{q}\) we can describe a double sequence \(x = (x_{k,l})\) by

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\[(x_{kl}) = \begin{cases} 1, & \text{if } k \in I_{r(p)}, l \in I_{s(q)}, (p, q = 1, 2, \ldots), \\ 0, & \text{otherwise}. \end{cases} \]

Therefore, \((x_{kl}) \in [C, 1]_{F^*N}\) and \((x_{kl}) \in st_2(FN)(X)\). On the other hand, \(x \notin [V, \vec{\lambda}]_{F^*N}(X)\).

Theorem 3 implies that \(x \notin FS_X(X)\); a contradiction and hence (1) must hold.

**Theorem 5.** Let \((X, \|\|)\) be an FNS and if \(\vec{\lambda} \in \Delta_2\) such that \(\lim_{r \to \infty} \frac{r_{rs}}{rs} = 1\). Then \(FS_X(X) \subset st_2(FN)(X)\).

Proof. Since \(\lim_{r \to \infty} \frac{r_{rs}}{rs} = 1\), we see following,

\[
\frac{1}{rs} \left\{ (k, l), k \leq r, l \leq s : (D\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon \right\} \leq \frac{1}{rs} \left\{ (k, l), k \leq r - \lambda, l \leq s - \mu_s : (D\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon \right\} + \frac{1}{rs} \left\{ (k, l) \in I_{r,s} : (D\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon \right\}
\]

\[
\leq \frac{r_{rs}}{rs} \lambda_{rs} + \frac{1}{rs} \left\{ (k, l) \in I_{r,s} : (D\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon \right\}
\]

\[
= \frac{r_{rs}}{rs} \lambda_{rs} + \frac{1}{\lambda_{rs}} \left\{ (k, l) \in I_{r,s} : (D\|x_{kl} - L\|, \tilde{0}) \geq \varepsilon \right\}
\]

for \(\varepsilon > 0\). This implies that \(x_{kl} \xrightarrow{st_2(FN)} L\), if \(x_{kl} \xrightarrow{FS_X} L\). Hence, \(FS_X(X) \subset st_2(FN)(X)\).

**Remark 1.** We could not show whether the condition \(\lim_{r \to \infty} \frac{r_{rs}}{rs} = 1\) in the above theorem is necessary. We left it as an open problem.

**Theorem 6.** \(FS_X - \lim_{k_{r,l} \to \infty} x_{kl} = L\) iff there is a subset \(K = \{(k_n, l_n) : k_1 < k_2 < \cdots ; l_1 < l_2 < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\delta_X(K) = 1\) and \(st_2(FN) - \lim_{r,s \to \infty} x_{kl,rs} = L\).

Proof. Assume that \(FS_X - \lim_{k_{r,l} \to \infty} x_{kl} = L\). Hence, for any \(\varepsilon > 0\) and \(s \in \mathbb{N}\), let

\[
K(s, \varepsilon) = \{(k, l) \in I_{r,s} : D(\|x_{kl} - L\|, \tilde{0}) \leq 1 - \frac{1}{s}\},
\]

\[
M(s, \varepsilon) = \{(k, l) \in I_{r,s} : D(\|x_{kl} - L\|, \tilde{0}) > 1 - \frac{1}{s}\}.
\]

Then \(\delta_X(K(s, \varepsilon)) = 0\) and

\[
M(1, \varepsilon) \supset M(2, \varepsilon) \supset \cdots \supset M(i, \varepsilon) \supset M(i + 1, \varepsilon) \supset \cdots
\]

and

\[
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\]
\[ \delta_T(M(s, \varepsilon)) = 1, \quad s = 1, 2, \ldots \]

Now we have to show for \((k, l) \in M(s, \varepsilon), x = (x_{k,l})\) is \(st_2(FN)\)-convergent to \(L\). Let \(x = (x_{k,l})\) is not \(st_2(FN)\)-convergent to \(L\) for some \((k, l) \in M(s, \varepsilon)\). Therefore, there is \(t > 0\) and are positive integers \(l_0, k_0\) such that \(D(\|x_{k,l} - L\|, 0) \leq 1 - t\) for all \(l \geq l_0, k \geq k_0\). And let \(\|x_{k,l} - L\|, 0) > 1 - t\) for all \(l \leq l_0, k \leq k_0\); hence,

\[ \delta_T \left( \left\{(k, l) \in I_{r,s} : D(\|x_{k,l} - L\|, 0) > 1 - t \right\} \right) = 0. \]

Since \(t > \frac{1}{s}\) we obtain

\[ \delta_T(M(s, \varepsilon)) = 0 \]

which contradicts (2). Finally, \(x = (x_{k,l})\) is double statistical convergent to \(L\).

Conversely, suppose that there is a subset \(K = \{(k_n, l_n) : k_1 < k_2 < \cdots ; l_1 < l_2 < \cdots \} \subseteq \mathbb{N} \times \mathbb{N}\) such that \(\delta_T(K) = 1\) and \(st_2(FN) - \lim_{r<s, \rightarrow \infty} x_{k,r,l_s} = L\). There exists \(N \in \mathbb{N}\) for every \(t \in (0, 1)\) and \(\varepsilon > 0\),

\[ \left( D(\|x_{k,l} - L\|, 0) > 1 - t, \text{ for all } k, l \geq N. \right. \]

Next,

\[ M(t, \varepsilon) = \left\{(k, l) \in I_{r,s} : \left( D(\|x_{k,l} - L\|, 0) \right) \leq 1 - t \right\} \subseteq \mathbb{N} \times \mathbb{N} - \{(k_{N+1}, L_{N+1}), (k_{N+2}, L_{N+2}), \ldots \}. \]

Therefore, \(\delta_T(M(s, \varepsilon)) \leq 1 - 1 = 0\). Hence, we have \(FS_T = \lim_{k,l \rightarrow \infty} x_{k,l} = L\).

**Definition 4.** Let \((X, \|\cdot\|)\) be an FNS, \(\bar{\lambda} \in \Delta_2\). The double sequence \(x = (x_{k,l})\) is called \(\bar{\lambda}\)-statistically Cauchy sequence with regards to fuzzy norm on \(X\) if there is a double subsequence \(x = \{x_{k_r,l_s}\}\) of \(x\) such that \((\bar{k}_r, \bar{l}_s) \in I_{r,s}\) for each \((r, s)\), \(FS_X = \lim_{r<s, \rightarrow \infty} x_{k_r,l_s} = L\) and for every \(\varepsilon > 0\)

\[ \lim_{r<s, \rightarrow \infty} \frac{1}{\bar{k}_r, \bar{l}_s} \left| \left\{(k, l) \in I_{r,s} : \left( D(\|x_{k,l} - x_{k_r,l_s}\|, 0) \right) > \varepsilon \right\} \right| = 0. \]

**Theorem 7.** The double sequence \(x = (x_{k,l})\) in \(X\) is \(\bar{\lambda}\)-statistically convergent with regards to fuzzy norm on \(X\) iff \(x = (x_{k,l})\) is \(\bar{\lambda}\)-statistically Cauchy sequence.

Proof. Let \(FS_X = \lim_{k,l \rightarrow \infty} x_{k,l} = L\) and

\[ K^{r,v} = \left\{(k, l) \in I_{r,s} : \left( D(\|x_{k,l} - L\|, 0) \right) < \frac{1}{r} \right\}, \]

We obtain the following

\[ K^{r+1,v+1} \subseteq K^{r,v} \quad \text{and} \quad \frac{\|K^{r+1,v+1} \cap I_{r,s}\|}{\bar{k}_r, \bar{l}_s} \rightarrow 1, \text{as } r, s \rightarrow \infty. \]

This implies that there is \(k_1\) and \(l_1\) such that \(r \geq k_1\) and \(s \geq l_1\) and \(\|K^{r+1,v+1} \cap I_{r,s}\| > 0\), that is, \(K^{1,1} \cap I_{r,s} \neq \emptyset\). We next choose \(k_2 \geq k_1\) and \(l_2 \geq l_1\) such that \(r \geq k_2\) and \(s \geq l_2\) implies that \(K^{2,2} \cap I_{r,s} \neq \emptyset\). Thus, for each pair \((r, s)\) such that \(k_1 \leq r < k_2\) and \(l_1 \leq s < l_2\). We select \((\bar{k}_r, \bar{l}_s) \in I_{r,s}\) such that \((\bar{k}_r, \bar{l}_s) \in K^{r,s} \cap I_{r,s}\) that is
(D \left( \|x_{k,l} - L\|, \hat{0} \right) < 1.

In general, we choose \( k_{t+1} \geq k_t \) and \( l_{v+1} \geq l_v \) such that \( r > k_{t+1} \) and \( s > l_{v+1} \). This implies \( K^{t+1,v+1} \cap I_{r,s} \neq \emptyset \). Thus, for all \((r,s)\) such that for \( k_t \leq r < k_{t+1} \) and \( l_v \leq s < l_{v+1} \) choose \((k_r, l_s) \in I_{r,s}\), that is,

\[
\left( D \left( \|x_{k,l} - L\|, \hat{0} \right) < 1 \right)
\]

Thus, \((k_r, l_s) \in I_{r,s}\) for each pair \((r,s)\) and

\[
\left( D \left( \|x_{k,l} - L\|, \hat{0} \right) < 1 \right)
\]

implies \( FS_X - \lim_{r,s \to \infty} x_{k_r,l_s} = L \). Also, for each \( \varepsilon > 0 \)

\[
\frac{1}{x_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k,l} - x_{k_r,l_s}\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

\[
\leq \frac{1}{x_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k,l} - L\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

\[
+ \frac{1}{x_{r,s}} \left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k_r,l_s} - L\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

Since \( FS_X - \lim_{k_{t+1} \to \infty} x_{k,t} = L \) and \( FS_X - \lim_{r,s \to \infty} x_{k_r,l_s} = L \), it follows that \( x \) is \( x \)-statistically Cauchy sequence.

Next, we assume \( x \) is an \( x \)-statistically Cauchy sequence. Then

\[
\left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k,l} - L\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

\[
\leq \left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k,l} - x_{k_r,l_s}\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

\[
+ \left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k_r,l_s} - L\|, \hat{0} \right) \geq \varepsilon \right\} \right|
\]

Therefore, \( FS_X - \lim_{k_{t+1} \to \infty} x_{k,t} = L \). Thus, the theorem is proved.

**Theorem 8.** \( FS_{\bar{x}}(X) \cap l_\infty^x (X) \) is a closed subset of \( l_2^x (X) \), if \( X \) is a fuzzy normed Banach space.

Proof. Let \( \left\{ x^n_n \right\}_{n \in \mathbb{N}} = \left( x^n_{k,l} \right)_{k,l \in \mathbb{N}} \) be a convergent sequence in \( FS_{\bar{x}}(X) \cap l_\infty^x (X) \) converging to \( x \in l_2^x (X) \). We have to prove that \( x \in FS_{\bar{x}}(X) \cap l_\infty^x (X) \). And let \( \left( x^n_{k,l} \right)_{k,l} \to L, \) for all \( n \in \mathbb{N} \). Taking a positive decreasing convergent sequence \( \left( \varepsilon_n \right)_{n \in \mathbb{N}}, \) where \( \varepsilon_n = \frac{\varepsilon}{2^n}, \) for a given \( \varepsilon > 0, \) we select \( \left( \varepsilon_n \right)_{n \in \mathbb{N}} \) which converges to zero. Choosing \( n \in \mathbb{Z}^+, \) such that \( \|x - x^n\|_\infty < \frac{\varepsilon}{4} \) for all \( n \geq r, s, \) then we get

\[
\left| \left\{ (k,l) \in I_{r,s} : D \left( \|x_{k,l} - L_n\|, \hat{0} \right) \geq \varepsilon_n \right\} \right| = 0
\]
and
\[ \lim_{r,s \to \infty} \frac{1}{\lambda_{r,s}} \left\{ (k,l) \in I_{r,s} : D \left( \| x_{k,l}^n - L_n \|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \right\} = 0. \]

Since
\[ \frac{1}{\lambda_{r,s}} \left\{ (k,l) \in I_{r,s} : D \left( \| x_{k,l}^n - L_n \|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \right\} = 0, \]

for all \( k, l \in \mathbb{N} \),
\[ \left\{ (k,l) \in I_{r,s} : D \left( \| x_{k,l}^n - L_n \|, \tilde{0} \right) \geq \frac{\varepsilon_n}{4} \right\} \cap \left\{ (k,l) \in I_{r,s} : D \left( \| x_{k,l}^{n+1} - L_{n+1} \|, \tilde{0} \right) \geq \frac{\varepsilon_{n+1}}{4} \right\} \]
is infinite. Hence, there must exist \((k,l) \in I_{r,s}\) for which we have, simultaneously,
\[ D \left( \| x_{k,l}^n - L_n \|, \tilde{0} \right) < \frac{\varepsilon_n}{4} \text{ and } D \left( \| x_{k,l}^{n+1} - L_{n+1} \|, \tilde{0} \right) < \frac{\varepsilon_{n+1}}{4}. \]

It follows that
\[ D \left( \| L_n - L_{n+1} \|, \tilde{0} \right) \leq D \left( \| L_n - x_{k,l}^n \|, \tilde{0} \right) \]
\[ + D \left( \| x_{k,l}^n - x_{k,l}^{n+1} \|, \tilde{0} \right) + D \left( \| x_{k,l}^{n+1} - L_{n+1} \|, \tilde{0} \right) \]
\[ \leq D \left( \| x_{k,l}^n - L_n \|, \tilde{0} \right) + D \left( \| x_{k,l}^{n+1} - L_{n+1} \|, \tilde{0} \right) \]
\[ + D \left( \| x - x^n \|_{\omega}, \tilde{0} \right) + D \left( \| x - x^{n+1} \|_{\omega}, \tilde{0} \right) \]
\[ \leq \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} + \frac{\varepsilon_n}{4} = \varepsilon_n. \]

It gives \((L_n)\) is a Cauchy sequence. We can write \( L_n \to L \in X \) as \( n \to \infty \), since \( X \) is a fuzzy normed Banach space. We prove that \( x_{k,l} \to L(FS_X) \). For any \( \varepsilon > 0 \), taking \( n \in \mathbb{N} \) such that \( \varepsilon_n < \frac{\varepsilon}{4} \),
\[ D \left( \| x_{k,l} - x_{k,l}^n \|_{\omega}, \tilde{0} \right) < \frac{\varepsilon}{4} \text{ and } D \left( \| L_n - L \|, \tilde{0} \right) < \frac{\varepsilon}{4}. \]

Hence, we have
\[
\frac{1}{\lambda r,s} \left| \left\{ (k,l) \in I_{r,s} : D\left( \left\| x_{k,l} - L \right\|_{\infty}, 0 \right) \geq \varepsilon \right\} \right|
\]

\[
\leq \frac{1}{\lambda r,s} \left| \left\{ (k,l) \in I_{r,s} : D\left( \left\| x_{k,l}^n - L_n \right\|, 0 \right) + D\left( \left\| L_n - L \right\|, 0 \right) \geq \varepsilon \right\} \right|
\]

\[
+ D\left( \left\| x_{k,l} - x_{k,l}^n \right\|_{\infty}, 0 \right) \]

\[
\leq \frac{1}{\lambda r,s} \left| \left\{ (k,l) \in I_{r,s} : D\left( \left\| x_{k,l}^n - L_n \right\|, 0 \right) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \geq \varepsilon \right\} \right|
\]

\[
\leq \frac{1}{\lambda r,s} \left| \left\{ (k,l) \in I_{r,s} : D\left( \left\| x_{k,l}^n - L_n \right\|, 0 \right) \geq \frac{\varepsilon}{2} \right\} \right| \to 0 \text{ as } r,s \to \infty.
\]

This gives \( x_{k,l} \to L(\mathcal{FS}_\lambda) \).

**Conclusion**

In this paper, we give \( \bar{\lambda} \)-statistically convergence, \( \bar{\lambda} \)-statistically Cauchy sequence, strongly \( \bar{\lambda} \)-summable for double sequences in fuzzy normed spaces. In further studies, the \( \bar{\lambda} \)-ideal convergence by using double sequences can be defined and examined in fuzzy normed spaces.

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**References**


