OSCILLATION PROPERTIES OF SOLUTIONS OF FRACTIONAL NEUTRAL DIFFERENTIAL EQUATIONS

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In this study, we consider a class of fractional neutral differential equations. We are going to give some new theorems that they complete and improve a number of results in the literature. Then we give an example to illustrating the main results.

Key words*: oscillation, neutral, delay, fractional derivative, fractional differential equations, oscillation theory*

1. Introduction

Fractional calculus is an interesting field of research due to its ability to describe complex nonlinear phenomena in chemistry, biology, physics, economics, engineering and other areas of science. Because of this, there have been many papers and books dealing with the theoretical development of fractional calculus and the solutions for nonlinear fractional differential equations [1-6].

Recently, there have been many studies concerning oscillation theory [7-25]. However, only a few papers consider the oscillation of fractional neutral differential equations [14,15]. Then, we strongly motivated by the research of Wang et al. [14] and Ganesan and Kumar [15]. They present some oscillation criteria for the fractional neutral differential equations.

In this study, we investigate the oscillatory behavior of the solutions of the following equations

$$D_t^{\alpha}\left(a(t)D_t^{\alpha}z(t)\right)+f\left(t,x(\sigma(t))\right)=0 \tag{1}$$

where $t\in[t_0,\infty)$, $t_0\in\mathbb{R}$, $D_t^{\alpha}(\cdot)$ denotes the modified Riemann-Liouville derivative [26], $z(t)=x(t)+p(t)x(\tau(t))$, $D_t^{2\alpha}p(t)\in C([t_0,\infty))$, $D_t^{\alpha}a(t)\in C([t_0,\infty))$, $\tau(t)\in C([t_0,\infty))$, $\sigma(t)\in C([t_0,\infty))$. Generally, we consider that the following conditions:

(H_1) $0\le p(t)\le p_0<\infty$ where p_0 is a constant;

(H_2) $\lim_{t\to+\infty}\tau(t)=+\infty$, $\lim_{t\to+\infty}\sigma(t)=+\infty$;

(H_3) $\tau\in C^1([t_0,\infty))$, $\tau'(t)\ge\tau_0>0$, $\tau\circ\sigma=\sigma\circ\tau$;

(H_4) $a(t)>0$ and $\lim_{t\to\infty}\int_{t_0}^{t}\frac{ds}{a(s)}<\infty$;

(H_5) $f(t,x)\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ satisfies $xf(t,x)>0$, for $x\ne0$ and there exists $q(t)\in C([t_0,\infty),\mathbb{R}^+)$ such that $\left|\frac{f(t,x)}{x}\right|\ge q(t)$;

(H_6) $1\le p(t)$ and eventually $p(t)\ne1$;

$$(H_7) \quad \frac{t}{\tau(t)} \geq l > 0;$$

and we consider the notation τ^{-1} for the inverse function of τ. The definition of modified Riemann-Liouville derivative and some of the key properties of its are listed as follows:

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} \left(f(\xi) - f(0) \right) d\xi, & 0 < \alpha < 1 \\ \left(f^{(n)}(t) \right)^{(\alpha-n)}, & 1 \leq n \leq \alpha \leq n+1 \end{cases}$$

$$D_t^{\alpha} \left(f(t) g(t) \right) = g(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} g(t)$$

$$D_t^{\alpha} f\left[g(t) \right] = f_g' \left[g(t) \right] D_t^{\alpha} g(t) = D_t^{\alpha} f\left[g(t) \right] \left(g'(t) \right)^{\alpha}$$

$$D_t^{\alpha} t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}$$

As usual, a solution $x(t)$ of (1) is termed oscillatory if it neither eventually positive nor eventually negative; otherwise, we call it non-oscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

2. Preliminary Lemmas

In this study, we consider following ξ variable transformation,

$$\xi = y(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad \xi_i = y(t_i) = \frac{t_i^{\alpha}}{\Gamma(1+\alpha)}, \quad i = 0,1$$

$$x(t) = \tilde{x}(\xi), a(t) = \tilde{a}(\xi), p(t) = \tilde{p}(\xi), q(t) = \tilde{q}(\xi).$$

Towards to $\tau(t)$, $\sigma(t)$, we have the next transformations.

Lemma 1. [14] Suppose (H_3) and (H_7) hold, we define the functions $\tilde{\tau}(\xi)$, $\tilde{\sigma}(\xi)$ as the following forms

$$\tilde{\tau}(\xi) = y\left(\tau\left(y^{-1}(\xi)\right)\right), \quad \tilde{\sigma}(\xi) = y\left(\sigma\left(y^{-1}(\xi)\right)\right)$$

then it satisfies

$$x\left(\tau(t)\right) = \tilde{x}\left(\tilde{\tau}(\xi)\right), \quad x\left(\sigma(t)\right) = \tilde{x}\left(\tilde{\sigma}(\xi)\right);$$

and a new condition

$$(H_3^*) : \tilde{\tau}'(\xi) \geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0 > 0, \quad \tilde{\tau} \circ \tilde{\sigma} = \tilde{\sigma} \circ \tilde{\tau}.$$

3. Main Results

For the sake of convenience, let us denote

$$R(\xi) = \int_{\xi_1}^{\xi} \tilde{a}^{-1}(s) ds, \quad S(\xi) = \int_{\xi}^{\infty} \tilde{a}^{-1}(s) ds$$

$$Q(\xi) = \min\left\{ \tilde{q}(\xi), \tilde{q}\left(\tilde{\tau}(\xi)\right) \right\}, \quad Q_*(\xi) = Q(\xi) \int_{\xi_1}^{\tilde{\eta}(\xi)} \tilde{a}^{-1}(s) ds, \quad Q^*(\xi) = Q(\xi) \int_{\tilde{\kappa}(\xi)}^{\infty} \tilde{a}^{-1}(s) ds.$$

Theorem 1. Assume that (H_1)-(H_5) hold, and let t_1 sufficiently large. Suppose that there exist two functions $\eta,\kappa\in C([t_0,\infty))$ such that $\eta(t)\le\sigma(t)\le\kappa(t)$ and $\lim_{t\to\infty}\eta(t)=\infty$. If the first-order neutral differential inequalities

$$\left(\tilde{u}(\xi)+\frac{p_0}{\tilde{\tau}_0}\tilde{u}(\tilde{\tau}(\xi))\right)'+Q_*(\xi)\tilde{u}(\tilde{\eta}(\xi))\le 0 \tag{2}$$

and

$$\left(\tilde{\phi}(\xi)+\frac{p_0}{\tilde{\tau}_0}\tilde{\phi}(\tilde{\tau}(\xi))\right)'-Q^*(\xi)\tilde{\phi}(\tilde{\kappa}(\xi))\ge 0 \tag{3}$$

where $\tilde{\tau}$ is defined in Lemma 1, have no positive solution, then Eq. (1) is oscillatory.

Proof. Assume that Eq. (1) has a non-oscillatory solution x. Then without loss of generality, we consider that x is eventually positive such that $x(t)>0$ on $[t_1,\infty)$, where t_1 is sufficiently large. It is equivalent to $\tilde{x}(\xi)>0$ on $[\xi_1,\infty)$, where ξ_1 is sufficiently large. Then let $\tilde{x}(\xi)>0$ on $[\xi_1,\infty)$. So, similarly to the proof of [14, 3.1. Theorem], we can obtain the following inequality

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'+\frac{p_0}{\tilde{\tau}_0}\left(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi))\right)'+Q(\xi)\tilde{z}(\sigma(\xi))\le 0. \tag{4}$$

From (1), one can see for sufficiently large ξ_1 and for all $\xi>\xi_1$,

$$\tilde{z}'(\xi)>0,\quad \left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'<0 \tag{5}$$

or

$$\tilde{z}'(\xi)<0,\quad \left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'<0. \tag{6}$$

Assume (5) holds. Using $\tilde{\eta}(\xi)<\tilde{\sigma}(\xi)$, we have

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'+\frac{p_0}{\tilde{\tau}_0}\left(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi))\right)'+Q(\xi)\tilde{z}(\tilde{\eta}(\xi))\le 0.$$

Now we denote $\tilde{u}(\xi)=\tilde{a}(\xi)\tilde{z}'(\xi)$. And from (5),

$$\tilde{z}(\xi)\ge\int_{\xi_1}^{\xi}\frac{\tilde{a}(s)\tilde{z}'(s)}{\tilde{a}(s)}ds\ge\tilde{a}(\xi)\tilde{z}'(\xi)\int_{\xi_1}^{\xi}\frac{1}{\tilde{a}(s)}ds=\tilde{u}(\xi)\int_{\xi_1}^{\xi}\frac{1}{\tilde{a}(s)}ds.$$

Then, we obtain that $\tilde{u}(\xi)$ is a positive solution of

$$\left(\tilde{u}(\xi)+\frac{p_0}{\tilde{\tau}_0}\tilde{u}(\tilde{\tau}(\xi))\right)'+Q_*(\xi)\tilde{u}(\tilde{\eta}(\xi))\le 0,$$

which contradicts our assumption that this inequality has no positive solutions. Now we consider the other case. So,

$$\int_{\xi}^{l}\tilde{z}'(s)ds\le\int_{\xi}^{l}\frac{\tilde{a}(s)\tilde{z}'(s)}{\tilde{a}(s)}ds\le\tilde{a}(\xi)\tilde{z}'(\xi)\int_{\xi}^{l}\frac{1}{\tilde{a}(s)}ds.$$

That is

$$\tilde{z}(l) \le \tilde{z}(\xi) + \tilde{a}(\xi)\tilde{z}'(\xi)\int_{\xi}^{l} \frac{1}{\tilde{a}(s)} ds.$$

For $l \to \infty$, we have that

$$-\tilde{a}(\xi)\tilde{z}'(\xi)S(\xi) \le \tilde{z}(\xi). \tag{7}$$

It follows from (4) and $\tilde{\sigma}(\xi) \le \tilde{\kappa}(\xi)$ that

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)' + \frac{p_0}{\tilde{\tau}_0}\left(\tilde{a}(\tilde{\tau}(\xi))\tilde{z}'(\tilde{\tau}(\xi))\right)' + Q(\xi)\tilde{z}(\tilde{\kappa}(\xi)) \le 0.$$

Then,

$$\left(\tilde{u}(\xi) + \frac{p_0}{\tilde{\tau}_0}\tilde{u}(\tilde{\tau}(\xi))\right)' - Q^*(\xi)\tilde{u}(\tilde{\kappa}(\xi)) \le 0,$$

and it can write the following form

$$-\left(\tilde{u}(\xi) + \frac{p_0}{\tilde{\tau}_0}\tilde{u}(\tilde{\tau}(\xi))\right)' - Q^*(\xi)\left(-\tilde{u}(\tilde{\kappa}(\xi))\right) \ge 0.$$

Now we denote $\tilde{\phi}(\xi) = -\tilde{u}(\xi)$. Then, $\tilde{\phi}(\xi)$ is a positive solution of

$$\left(\tilde{\phi}(\xi) + \frac{p_0}{\tilde{\tau}_0}\tilde{\phi}(\tilde{\tau}(\xi))\right)' - Q^*(\xi)\tilde{\phi}(\tilde{\kappa}(\xi)) \ge 0$$

which is a contradiction and the proof is complete.

Theorem 2. Assume that (H_2)-(H_6) hold, and there exist $\delta_1, \delta_2, \mu_1, \mu_2$ continuous functions such that $\delta_1(t) \le t \le \delta_2(t)$, $\mu_1(t) \le t \le \mu_2(t)$, $\tau(\delta_1(t)) \le t \le \tau(\delta_2(t))$, $\tau(\mu_1(t)) \le \sigma(t) \le \tau(\mu_2(t))$, and $\lim_{t\to\infty} \mu_1(t) = \lim_{t\to\infty} \delta_1(t) = \infty$. Finally, assume that, for sufficiently large $t \ge t_1$,

$$\int_{\xi}^{\infty} \tilde{q}(s)\tilde{p}_1(\tilde{\sigma}(s))ds = \infty \tag{8}$$

and

$$\int_{\xi}^{\infty} \tilde{q}(s)\tilde{p}_2(\tilde{\sigma}(s))S(\tilde{\mu}_2(s))ds = \infty, \tag{9}$$

where

$$\tilde{p}_1(s) = \frac{1}{\tilde{p}(\tilde{\tau}^{-1}(s))}\left(1 - \frac{R\left(\tilde{\tau}^{-1}(\tilde{\delta}_2(s))\right)}{R\left(\tilde{\tau}^{-1}(s)\right)}\frac{1}{\tilde{p}\left(\tilde{\tau}^{-1}(\tilde{\tau}^{-1}(s))\right)}\right)$$

and

$$\tilde{p}_2(s) = \frac{1}{\tilde{p}(\tilde{\tau}^{-1}(s))}\left(1 - \frac{S\left(\tilde{\tau}^{-1}(\tilde{\delta}_1(s))\right)}{S\left(\tilde{\tau}^{-1}(s)\right)}\frac{1}{\tilde{p}\left(\tilde{\tau}^{-1}(\tilde{\tau}^{-1}(s))\right)}\right).$$

Then Eq. (1) is oscillatory.

Proof. Let $x(t)$ is an eventullay positive solution of Eq. (1). Then, $D_t^{\alpha} z(t)$ does not change sign eventually.

Case I. Suppose that $D_t^{\alpha} z(t) > 0$. Hence,

$$\begin{aligned} x(t) &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right) \\ &= \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - \frac{z(\tau^{-1}(\tau^{-1}(t))) - x(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\geq \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - \frac{z(\tau^{-1}(\tau^{-1}(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right) \\ &\geq \frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t)) - \frac{z(\tau^{-1}(\delta_2(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right). \end{aligned}$$

Using that $\tilde{a}(\xi)\tilde{z}'(\xi)$ is strictly decreasing, we obtain

$$\tilde{z}(\xi) \geq \int_{\xi_1}^{\xi} \left(\tilde{a}(s)\tilde{z}'(s)\right) ds \geq \tilde{a}(\xi)\tilde{z}'(\xi)\int_{\xi_1}^{\xi} \tilde{a}^{-1}(s)\, ds.$$

As a result,

$$\left(\frac{\tilde{z}(\xi)}{R(\xi)}\right)' \leq 0.$$

Using the above last inequality and $\tilde{\delta}_2(\xi) \geq \xi$, we have that

$$\tilde{z}\left(\tilde{\tau}^{-1}(\tilde{\delta}_2(\xi))\right) \leq \frac{R\left(\tilde{\tau}^{-1}(\tilde{\delta}_2(\xi))\right)}{R\left(\tilde{\tau}^{-1}(\xi)\right)} \tilde{z}\left(\tilde{\tau}^{-1}(\xi)\right).$$

Thus

$$\tilde{x}(\xi) \geq \frac{\tilde{z}\left(\tilde{\tau}^{-1}(\xi)\right)}{\tilde{p}\left(\tilde{\tau}^{-1}(\xi)\right)}\left(1 - \frac{R\left(\tilde{\tau}^{-1}(\tilde{\delta}_2(\xi))\right)}{R\left(\tilde{\tau}^{-1}(\xi)\right)} \frac{1}{\tilde{p}\left(\tilde{\tau}^{-1}(\tilde{\tau}^{-1}(\xi))\right)}\right) = \tilde{p}_1(\xi)\tilde{z}\left(\tilde{\tau}^{-1}(\xi)\right).$$

From Eq. (1), we have

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)' + \tilde{q}(\xi)\tilde{x}\left(\tilde{\sigma}(\xi)\right) \leq 0$$

and so

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)' + \tilde{q}(\xi)\tilde{p}_1\left(\tilde{\sigma}(\xi)\right)\tilde{z}\left(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))\right) \leq 0.$$

Then we obtain

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)' + \tilde{q}(\xi)\tilde{p}_1\left(\tilde{\sigma}(\xi)\right)\tilde{z}\left(\tilde{\mu}_1(\xi)\right) \leq 0. \tag{10}$$

Using $\tilde{u}(\xi) = \tilde{a}(\xi)\tilde{z}'(\xi)$ in (10), we obtain $\tilde{u}(\xi)$ is a positive solution of

$$\tilde{u}'(\xi) + \tilde{q}(\xi)\tilde{p}_1\left(\tilde{\sigma}(\xi)\right)R\left(\tilde{\mu}_1(\xi)\right)\tilde{u}\left(\tilde{\mu}_1(\xi)\right) \leq 0.$$

Then [23, Theorem 1] point out that following differential equation

$$\tilde{u}'(\xi)+\tilde{q}(\xi)\tilde{p}_1(\tilde{\sigma}(\xi))R(\tilde{\mu}_1(\xi))\tilde{u}(\tilde{\mu}_1(\xi))=0 \tag{11}$$

has a positive solution. Nevertheless, from [21, Theorem 2] with (8), we have that Eq. (11) cannot have positive solutions. This is a contradiction.

Case II. Now, we consider the other case $D_t^{\alpha}z(t)<0$. Using similar operations in Case I, we obtain

$$x(t)\geq\frac{1}{p(\tau^{-1}(t))}\left(z(\tau^{-1}(t))-\frac{z(\tau^{-1}(\delta_1(t)))}{p(\tau^{-1}(\tau^{-1}(t)))}\right).$$

And (7) implies that

$$\left(\frac{\tilde{z}(\xi)}{S(\xi)}\right)'\geq 0.$$

For $\tilde{\delta}_1(\xi)\leq\xi$, we have

$$\begin{aligned}\tilde{x}(\xi)&\geq\frac{\tilde{z}(\tilde{\tau}^{-1}(\xi))}{\tilde{p}(\tilde{\tau}^{-1}(\xi))}\left(1-\frac{S(\tilde{\tau}^{-1}(\tilde{\delta}_1(\xi)))}{S(\tilde{\tau}^{-1}(\xi))}\frac{1}{\tilde{p}(\tilde{\tau}^{-1}(\tilde{\tau}^{-1}(\xi)))}\right)\\&=\tilde{p}_2(\xi)\tilde{z}(\tilde{\tau}^{-1}(\xi)).\end{aligned}$$

Then,

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'+\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))\tilde{z}(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi)))\leq 0.$$

Using that $\tilde{z}(\xi)$ is strictly decreasing, we obtain

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'+\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))\tilde{z}(\tilde{\mu}_2(\xi))\leq 0.$$

From (7), we have that

$$\left(\tilde{a}(\xi)\tilde{z}'(\xi)\right)'-\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))S(\tilde{\mu}_2(\xi))\tilde{a}(\tilde{\mu}_2(\xi))\tilde{z}'(\tilde{\mu}_2(\xi))\leq 0.$$

Using $\tilde{u}(\xi)=\tilde{a}(\xi)\tilde{z}'(\xi)$, then

$$\tilde{u}'(\xi)-\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))S(\tilde{\mu}_2(\xi))\tilde{u}'(\tilde{\mu}_2(\xi))\leq 0.$$

For $\tilde{y}(\xi)=-\tilde{u}(\xi)$. $\tilde{y}(\xi)$ is a positive solution of following differential inequality

$$\tilde{y}'(\xi)-\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))S(\tilde{\mu}_2(\xi))\tilde{y}(\tilde{\mu}_2(\xi))\geq 0.$$

Hence, from [27, Lemma 2.3]

$$\tilde{y}'(\xi)-\tilde{q}(\xi)\tilde{p}_2(\tilde{\sigma}(\xi))S(\tilde{\mu}_2(\xi))\tilde{y}(\tilde{\mu}_2(\xi))=0 \tag{12}$$

has a positive solution. Nevertheless, from [21, Theorem 2] with (9), we have that Eq. (12) cannot have any positive solutions. This is a contradiction. So, the proof is complete.

Combining Theorem 2 with the oscillation criteria presented in Ladde et al. [22, Theorem 2.1.1 and Theorem 2.4.1], we can give the following corollary.

Corollary 1. Under the conditions of Theorem 2, if

$$\liminf_{\xi\to\infty}\int_{\tilde{\mu}_1(\xi)}^{\xi}\tilde{q}(s)\,\tilde{p}_1(\tilde{\sigma}(s))\,ds>\frac{1}{e}$$

and

$$\liminf_{\xi\to\infty}\int_{\xi}^{\tilde{\mu}_2(\xi)}\tilde{q}(s)\,\tilde{p}_2(\tilde{\sigma}(s))\,S(\tilde{\mu}_2(s))\,ds>\frac{1}{e}.$$

Then Eq. (1) is oscillatory.

4. An Illustrative Example

We present a simple example to illustrate the obtained results. Consider the following equation

$$D_t^{1/2}\left(e^{\sqrt{t}/\Gamma(3/2)}D_t^{1/2}\left(x(t)+2x(t/4)\right)\right)+tx(t/16)=0,\ \ t\geq 1. \tag{13}$$

In (1), we set $a(t)=e^{\sqrt{t}/\Gamma(3/2)}$, $p(t)=2$, $f(t,x(\sigma(t)))=tx(t/16)$, $q(t)=t$, $\tau(t)=\frac{t}{4}$, $\sigma(t)=\frac{t}{16}$, $\alpha=\frac{1}{2}$. Then using a variable substitution we have,

$$\xi=y(t)=\frac{t^{1/2}}{\Gamma(3/2)},\ y^{-1}(\xi)=\Gamma^2\left(\frac{3}{2}\right)\xi^2,\ \xi_1=\frac{1}{\Gamma(3/2)}.$$

Then

$$\tilde{a}(\xi)=a\left(y^{-1}(\xi)\right)=e^{\xi}$$

$$\tilde{\sigma}(\xi)=y\left(\sigma\left(y^{-1}(\xi)\right)\right)=\frac{\xi}{4}$$

$$\tilde{\tau}(\xi)=y\left(\tau\left(y^{-1}(\xi)\right)\right)=\frac{\xi}{2},\ \tilde{\tau}'(\xi)=\frac{1}{2}=\tilde{\tau}_0>0 \text{ and } \tilde{\tau}^{-1}(\xi)=2\xi$$

$$\tilde{q}(\xi)=\xi^2\Gamma^2(3/2)$$

Choosing $\delta_1(t)=t$, $\delta_2(t)=4t$, $\mu_1=\frac{t}{4}$ and $\mu_2(t)=2t$, furthermore, we have

$$\tilde{\delta}_1(\xi)=y\left(\delta_1\left(y^{-1}(\xi)\right)\right)=\xi$$

and similarly $\tilde{\delta}_2(\xi)=2\xi$, $\tilde{\mu}_1(\xi)=\frac{\xi}{2}$, $\tilde{\mu}_2(\xi)=\sqrt{2}\xi$. Then we obtain

$$\liminf_{\xi\to\infty}\int_{\tilde{\mu}_1(\xi)}^{\xi}\tilde{q}(s)\,\tilde{p}_1(\tilde{\sigma}(s))\,ds=\liminf_{\xi\to\infty}\int_{\xi/2}^{\xi}\frac{\Gamma^2(3/2)}{2}s^2\left(1-\frac{R(s)}{2R(s/2)}\right)ds=\infty>\frac{1}{e}$$

and

$$\liminf_{\xi\to\infty}\int_{\xi}^{\tilde{\mu}_2(\xi)}\tilde{q}(s)\,\tilde{p}_2(\tilde{\sigma}(s))\,S(\tilde{\mu}_2(s))\,ds=\liminf_{\xi\to\infty}\int_{\xi}^{\sqrt{2}\xi}\frac{\Gamma^2(3/2)}{4}s^2S\left(\sqrt{2}s\right)ds=\infty>\frac{1}{e}$$

So we conclude that (13) is oscillatory by Corollary 1.

5. Conclusion

In this work, we are concern with the oscillatory solutions of fractional neutral differential equations. Based on a variable transformation, we obtain some oscillation theorems for the equation.

The obtained theorems complete and improve the many oscillation results. Because of the condition (H_4), the Eq. (11) gives us the importance of this work.

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