CHEBYSHEV WAVELET COLLOCATION METHOD FOR GINZBURG-LANDAU EQUATION

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The main aim of this paper is to investigate the efficient Chebyshev wavelet collocation method for Ginzburg-Landau equation. The basic idea of this method is to have the approximation of Chebyshev wavelet series of a non-linear partial differential equation. We demonstrate how to use the method for the numerical solution of the Ginzburg-Landau Equation with initial and boundary conditions. For this purpose, we have obtained operational matrix for Chebyshev wavelets. By applying this technique in Ginzburg-Landau equation, the partial differential equation is converted into an algebraic system of non-linear equations and this system has been solved using Maple computer algebra system. We demonstrate the validity and applicability of this technique which has been clarified by using an example. Exact solution is compared with an approximate solution. Moreover, Chebyshev wavelet collocation method is found to be acceptable, efficient, accurate and computationall for the non-linear or linear partial differential equation.

Key Words: Chebyshev wavelet collocation method, Ginzburg-Landau Equation, operational matrices, Non-linear PDE.

1. Introduction

When viewed from a historical perspective, wavelet analysis is a new method, although based on the work of the 19th century mathematical foundation Joseph Fourier. Fourier has laid the foundations for the theory of frequency analysis and proved to be very important and effective. This approach is suitable for signal filtering and compression with time-dependent waveform-like properties. However, it may not be suitable for functions with regional characteristics. In other words, series of Fourier gives only frequency resolution but not time resolution. Although we can identify all the frequencies in a signal, we can not determine when those frequencies are. Therefore, the researchers' interest has become increasingly scale-based from frequency-based analysis.

The concept of wavelet was first dealt with in 1909 by Alfred Haar’s master thesis. Since that time, Wavelet analysis methods have been developed. After that, research on wavelets has reached international dimension by Y.Meyer and colleagues. The main algorithm is based on the work of Stephane Mallat in 1988. At the beginning of this researches are Ingrid Daubechies, Ronald Coifman and Victor Wickerhauser, who pioneered others. The working and application areas of wavelets are growing very rapidly.

In general, wavelets have been very used successful in many different fields of science and engineering such as thermal science, signal analysis, data compression, seismology. Wavelets can be used for algebraic manipulations in the system of equations obtained which leads to better resulting system. For this reason, wavelet methods have become a more preferred numerical method today.

Wavelets have many different families. These are the most familiar and simple Haar wavelets. Haar wavelets are used by many researchers because of their simplicity [17-22]. The disadvantage of using Haar's functions is the low accuracy of the numerical approach. Therefore, it is thought that smooth Chebyshev wavelets have a more accurate approach to Haar wavelets [3]. Chebyshev polynomials and their properties are used to obtain the general procedure for the formation of matrices. Then, operational matrices of Chebyshev wavelet expansions are applied for the solution of non-linear partial differential equations. Due to their smoothness and good interpolation properties, the correctness of Chebyshev wavelets is better than that of Haar wavelets. and therefore we use the chebyshev wavelet collocation method to solve Ginzburg-Landau equation.

The complex Ginzburg-Landau equation is known as the first model of the amplitude equations and is given by:

$$\frac{\partial A}{\partial t} = A - (1 + i\alpha)|A|^2 A + (1 + i\beta)\nabla^2 A \tag{1}$$

In the complex Ginzburg Landau equation, the α and β parameters are real parameters, and $A(x,t)$ is a complex scalar domain called the order parameter. The qualitative dynamic behavioral solutions of equality change with coefficients α and β. These coefficients can be obtained from the equations found at the end as a result of exhaustive calculations for a given system.

For large values of $|\alpha|$ and $|\beta|$, the complex Ginzburg Landau equation is reduced to the non-linear Schöredinger equation (Gabbay, 1997). In recent years, the dynamics outside such boundaries have been working intensively in theory. The complex Ginzburg Landau equation shows a broadly valid behavior. The equation in two dimensions has spatial wave solutions and is perhaps the simplest equation solved in this form. The time dependent Ginzburg Landau Theory is used to determine the non - equilibrium properties of superconductors. More elementary and detailed introductions to the concepts underlying the equation can be found in Manneville (1990), van Saarloos (1995), van Hecke et al. (1994), Nicolis (1995), and Walgraaf (1997).

Lately, the Ginzburg-Landau Equation has been numerically solved by using Galerkin finite element method [23], modified simple equation method [24] and decomposition method [14].

In this paper, we have discussed the Chebyshev wavelet collocation method and we have solved the Ginzburg-Landau equation and examined the accuracy of the results.

2. Properties of Chebyshev Wavelets and Its Operational Matrix of Integration

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet $\varphi(x)$. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets as [1]

$$\varphi_{a,b}(x) = |a|^{-\frac{1}{2}} \varphi\left(\frac{x-b}{a}\right), \qquad a,b \in \mathbb{R}, a \neq 0 \tag{2}$$

Chebyshev wavelets are written as $\varphi_{n,m}(x) = \varphi(k,n,m,x)$, have four arguments, where $k = 0,1,2,\ldots$ and $n = 1,2,\ldots,2^{k-1}$, m is degree of Chebyshev polynomials of the first kind, and x denotes the normalized time. They are defined on the interval $[0,1]$ as

$$\varphi_{n,m}(x)=\begin{cases}\dfrac{\alpha_m 2^{k/2}}{\sqrt{\pi}}T_m\left(2^k x-2n+1\right) & , \dfrac{n-1}{2^{k-1}}\le x\le\dfrac{n}{2^{k-1}} \\ 0 & , otherwise\end{cases} \tag{3}$$

where

$$\alpha_m=\begin{cases}\sqrt{2} & m=0 \\ 2 & m=1,2,\dots\end{cases} \tag{4}$$

and $m=0,1,..,M-1$, $n=1,2,\dots,2^{k-1}$. Here, $T_m(x)$ are the well-known Chebyshev polynomials of order m, which are orthogonal with respect to the weight function $W(x)=1/\sqrt{1-x^2}$ on the interval $[-1,1]$, and satisfy the following recursive formula:

$$T_0(x)=1 \ , \ T_1(x)=x \ , \ T_{m+1}(x)=2xT_m(x)-T_{m-1}(x) \tag{5}$$

We should note that $T_m\left(2^{k+1}x-2n+1\right)$ are Chebyshev polynomials of the first kind of degree m that are orthogonal with respect to the weight function $W_n(x)=W\left(2^{k+1}x-2n+1\right)$. A function $f(x)\in L_W^2[0,1]$ may be expanded into Chebyshev wavelets basis as

$$f(x)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\kappa_{n,m}\varphi_{n,m}(x) \tag{6}$$

where

$$\kappa_{n,m}=\left\langle f(x),\varphi_{n,m}(x)\right\rangle \tag{7}$$

$\langle .,.\rangle$ denotes the inner product with weight function $W_n(x)$ in equation (7). Truncated form of equation (6) can be written as

$$f(x)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\kappa_{n,m}\varphi_{n,m}(x)=\mathrm{K}^T\Psi(x)\,. \tag{8}$$

$$\mathrm{K}=\left[\kappa_{10},\kappa_{11},\dots,\kappa_{1M-1},\kappa_{20},\dots,\kappa_{2M-1},\dots,\kappa_{2^{k-1}0},\dots,\kappa_{2^{k-1}M-1}\right]^T \tag{9}$$

and

$$\Psi(x)=\left[\varphi_{10}(x),\varphi_{11}(x),\dots,\varphi_{1M-1}(x),\varphi_{20}(x),\dots,\varphi_{2M-1}(x),\dots,\varphi_{2^{k-1}0}(x),\dots,\varphi_{2^{k-1}M-1}(x)\right]^T \tag{10}$$

For simplicity, we write (9)

$$\mathrm{K}=\left[\kappa_1,\kappa_2,\dots,\kappa_{2^{k-1}}\right]^T,\ \Psi(x)=\left[\varphi_1,\varphi_2,\dots,\varphi_{2^{k-1}}\right]^T$$

and

$$\kappa_i=\left[\kappa_{i0},\kappa_{i1},\dots,\kappa_{i,M-1}\right]^T,\ \psi_i(x)=\left[\varphi_{i0}(x),\varphi_{i1}(x),\dots,\varphi_{i,M-1}(x)\right]^T \ \text{for } i=1,2,\dots,2^{k-1} \tag{11}$$

Moreover, an arbitrary function of two variables $u(x,t)$ defined on $[0,1]\times[0,1]$, can be explained into Chebyshev wavelet basis as:

$$u(x,t)\cong\sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{k-1}M}\kappa_{ij}\varphi_i(x)\varphi_j(t)=\Psi(t)^T\,\mathrm{K}\Psi(x) \tag{12}$$

where $\mathrm{K}=\left[\kappa_{ij}\right]$ and $\kappa_{ij}=<\varphi_i(x),<u(x,t),\varphi_j(t)\gg$. Taking the collocation points

$$t_i=\frac{2i-1}{2^k M} \quad , \quad i=1,2,\dots,2^{k-1}M$$

we define the $2^{k-1}M\times2^{k-1}M$ matrix Φ as:

$$\Phi=\left[\Psi\left(\frac{1}{2^k M}\right),\Psi\left(\frac{3}{2^k M}\right),\dots,\Psi\left(\frac{2^k M-1}{2^k M}\right)\right] \tag{13}$$

We need to know how to calculate the integral of the operational matrix in order to implement our method. That's why, Kilicman and Al Zhour [13] investigated the generalized integral operational matrix. The integral of the vector $\Psi(x)$ can be represented as

$$\int_0^x \Psi(\tau)\,d\tau \cong P\Psi(x) \tag{14}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix of the one-time integral of $\Psi(x)$ [13]. Moreover, Kilicman and Al Zhour [13] generalized integral operational matrices P^n of n-times integration of $\Psi(x)$ can showed as follows:

$$\underbrace{\int_0^x \dots \int_0^x \Psi(\tau)d\tau \dots d\tau}_{n-times} \simeq P^n\Psi(x) \tag{15}$$

Çelik [4] expressed a uniform method to obtain the corresponding integral operational matrix of different basis. Therefore, the operational matrix of $\Psi(x)$ can showed following form:

$$P = \Phi P_B \Phi^{-1} \tag{16}$$

and in generally,

$$P^n = \Phi P_B^{\,n} \Phi^{-1} \tag{17}$$

where $P_B^{\,n}$ is the operational matrix of n-time integral of the block pulse functions

$$P_B^n = \frac{1}{m^n}\frac{1}{(n+1)!}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \ddots & \xi_{m-3} \\ 0 & 0 & 0 & 1 & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

where $\xi_i = (i+1)^{n+1} - 2i^{n+1} + (i-1)^{n+1}$.

3. Numerical Method

Let's assume α and β are constant and equal to zero in equation (1). We consider the following Ginzburg Landau equation described by

$$\frac{\partial u}{\partial t} - u - |u|^2 u - \frac{\partial^2 u}{\partial x^2} = x\left(1 - t + x^2 t^3\right) \tag{19}$$

with initial condition and boundary conditions

$$u(x,0) = 0 \tag{20}$$

$$u(0,t) = 0 \tag{21}$$

$$u(1,t) = t \tag{22}$$

The exact solution of this problem $u(x,t) = xt$ [14].

Now, let's solve this problem using the Chebyshev wavelet collocation method with $k = 2, M = 3$,

$$\Psi(x) = \left[\varphi_{10}(x), \varphi_{11}(x), \varphi_{12}(x), \varphi_{20}(x), \varphi_{21}(x), \varphi_{22}(x)\right] \tag{23}$$

obtain and we have

$$\varphi_{10}(x) = \begin{cases} \frac{2}{\sqrt{\pi}} T_0(4x-1) & , \ 0 \le x \le \frac{1}{2} \\ 0 & ,otherwise \end{cases} \qquad \varphi_{12}(x) = \begin{cases} \frac{2\sqrt{2}}{\sqrt{\pi}} T_2(4x-1) & , \ 0 \le x \le \frac{1}{2} \\ 0 & ,otherwise \end{cases}$$

$$\varphi_{11}(x)=\begin{cases}\dfrac{2\sqrt{2}}{\sqrt{\pi}}T_1(4x-1) & , \ 0\le x\le \dfrac{1}{2}\\ 0 & ,otherwise\end{cases}\qquad \varphi_{20}(x)=\begin{cases}\dfrac{2}{\sqrt{\pi}}T_0(4x-3) & , \ \dfrac{1}{2}\le x\le 1\\ 0 & ,otherwise\end{cases}$$

$$\varphi_{21}(x)=\begin{cases}\dfrac{2\sqrt{2}}{\sqrt{\pi}}T_1(4x-3) & , \ \dfrac{1}{2}\le x\le 1\\ 0 & ,otherwise\end{cases}\qquad \varphi_{22}(x)=\begin{cases}\dfrac{2\sqrt{2}}{\sqrt{\pi}}T_2(4x-3) & , \ \dfrac{1}{2}\le x\le 1\\ 0 & ,otherwise\end{cases}$$

we assume

$$\frac{\partial^3 u}{\partial t\partial x^2}=\Psi(x)^T\,\mathrm{K}\Psi(t) \tag{24}$$

Where $\mathrm{K}=\left[\kappa_{ij}\right]_{m\times m}$ is an unknown matrix which should be found and $\Psi(t)$ is the vector that defined in (10). By integration of (24) one time with respect to t and considering (20), we obtain:

$$u_{xx}(x,t)=\Psi(x)^T\,\mathrm{K}P\Psi(t)+u_{xx}(x,0) \tag{25}$$

Also by integration of (24) two times with respect to x we get:

$$u_t(x,t)=\Psi(x)^T\left(P^2\right)^T\mathrm{K}\Psi(t)+u_t(0,t)+xu_{tx}(0,t) \tag{26}$$

by putting $x=1$ into (26).

$$u_t(1,t)=\Psi(1)^T\left(P^2\right)^T\mathrm{K}\Psi(t)+u_t(0,t)+u_{xt}(0,t) \tag{27}$$

and (27) substitute in (26)

$$u_t(x,t)=\Psi(x)^T\left(P^2\right)^T\mathrm{K}\Psi(t)+u_t(0,t)+x\Big(u_t(1,t)-u_t(0,t)-\Psi(1)^T\left(P^2\right)^T\mathrm{K}\Psi(t)\Big) \tag{28}$$

Now by integration of (26) one times with respect to t we get

$$\begin{aligned}&u(x,t)=\Psi(x)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)-x\Psi(1)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)+G(x,t)\\&G(x,t)=u(x,0)+\big(u(0,t)-u(0,0)\big)+x\Big\{\big[u(1,t)-u(1,0)\big]+\big[u(0,t)-u(0,0)\big]\Big\}\end{aligned} \tag{29}$$

Now by replacing (25), (29) and (28) into (19);

$$\begin{aligned}&\Psi(x)^T\left(P^2\right)^T\mathrm{K}\Psi(t)-x\Psi(1)^T\left(P^2\right)^T\mathrm{K}\Psi(t)+x-\\&\left[\Psi(x)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)-x\Psi(1)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)+xt\right]-\\&\left\{\begin{array}{l}\left|\Psi(x)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)-x\Psi(1)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)+xt\right|^2\\ .\left[\Psi(x)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)-x\Psi(1)^T\left(P^2\right)^T\mathrm{K}P\Psi(t)+xt\right]\end{array}\right\}-\Psi(x)^T\,\mathrm{K}P\Psi(t)=x\left(1-t+x^2t^3\right)\end{aligned}$$

(30)

We have the graphic and error table for the approximate solution of this equation for $k=2, M=3$ is as follows:

Figure 1: Ginzburg-Landau Equation Exact Solution

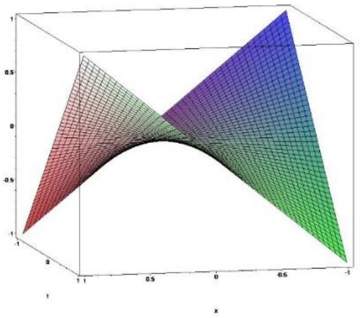

Figure 2: CWCM Approximate Solution for $k = 2, M = 3$

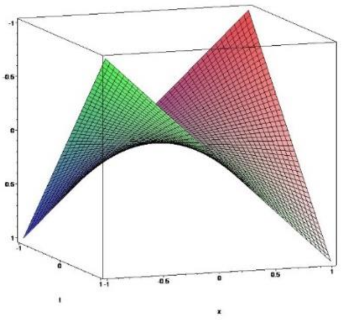

Figure 3: CWCM Error Analysis for $k = 2, M = 3$

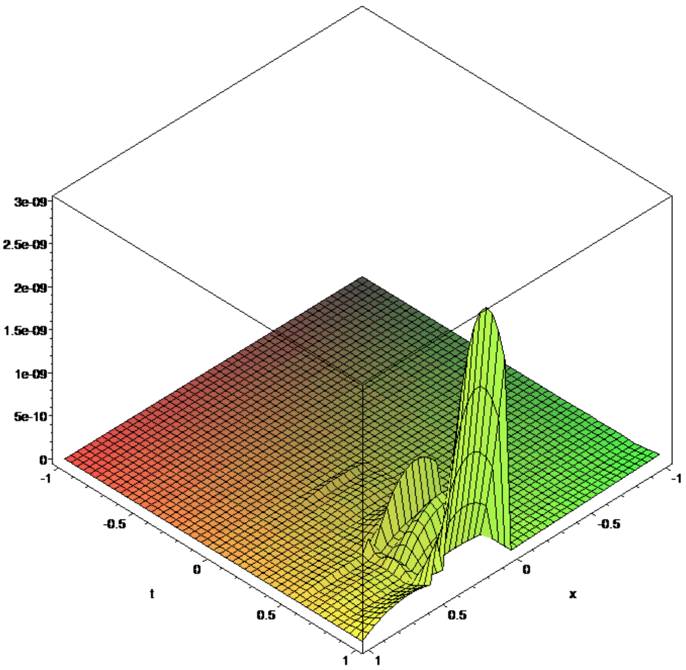

Table 1: Error Table for $k = 2, M = 3$

x	t	$\lvert u_{numeric} - u_{exact} \rvert$	x	t	$\lvert u_{numeric} - u_{exact} \rvert$
	0.08333333333	$0.681792296230998574 \times 10^{-11}$		0.08333333333	$0.170033431778904287 \times 10^{-11}$
0.08333333333	0.2500000000	$0.316179825377016500 \times 10^{-10}$	0.5833333333	0.2500000000	$0.307129321974741744 \times 10^{-11}$
	0.4166666667	$0.100399584479493598 \times 10^{-9}$		0.4166666667	$0.879699091349550600 \times 10^{-11}$
	0.5833333333	$0.184055888585721306 \times 10^{-9}$		0.5833333333	$0.522898391253079354 \times 10^{-11}$
	0.7500000000	$0.433789712483978463 \times 10^{-9}$		0.7500000000	$0.280424017340408227 \times 10^{-10}$
	0.9166666667	$0.107897452084060320 \times 10^{-8}$		0.9166666667	$0.819099232884923368 \times 10^{-10}$
	0.08333333333	$0.903849564637404513 \times 10^{-11}$		0.08333333333	$0.189202820077838396 \times 10^{-11}$
0.2500000000	0.2500000000	$0.368135105732747548 \times 10^{-10}$	0.7500000000	0.2500000000	$0.322461501944815155 \times 10^{-11}$
	0.4166666667	$0.113826004177752793 \times 10^{-9}$		0.4166666667	$0.939287536638744314 \times 10^{-11}$
	0.5833333333	$0.191916593728080898 \times 10^{-9}$		0.5833333333	
	0.7500000000	$0.465085248091412496 \times 10^{-9}$		0.7500000000	$0.295716784393107446 \times 10^{-10}$
	0.9166666667	$0.117287252132669550 \times 10^{-8}$		0.9166666667	$0.868570770862220344 \times 10^{-10}$
	0.08333333333	$0.432351376922213149 \times 10^{-11}$		0.08333333333	$0.921290821409570526 \times 10^{-12}$
0.4166666667	0.2500000000	$0.145935763473659108 \times 10^{-10}$	0.9166666667	0.2500000000	$0.153549395420782276 \times 10^{-11}$
	0.4166666667	$0.436417568749902784 \times 10^{-10}$		0.4166666667	$0.450250947636732235 \times 10^{-11}$
	0.5833333333	$0.637907782152780100 \times 10^{-10}$		0.5833333333	$0.182431847406405723 \times 10^{-11}$
	0.7500000000	$0.165010338726290230 \times 10^{-9}$		0.7500000000	$0.141059386393749264 \times 10^{-10}$
	0.9166666667	$0.426662261077126460 \times 10^{-9}$		0.9166666667	$0.415226741878882422 \times 10^{-10}$

Similarly, We have the graphic and error table for the approximate solution of this equation for $k=2, M=4$ is as follows:

Figure 4: CWCM Approximate solution for $k=2, M=4$

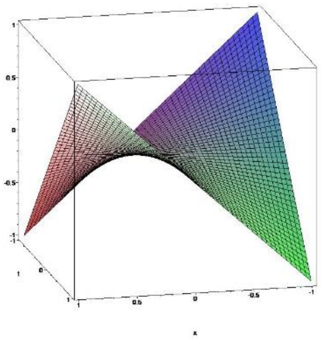

Figure 5: CWCM Error Analysis for $k=2, M=4$

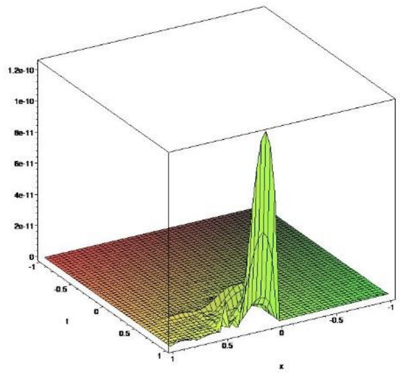

Table 2: Error Table for $k=2, M=4$

x	t	$\lvert u_{numeric} - u_{exact} \rvert$	x	t	$\lvert u_{numeric} - u_{exact} \rvert$
0.0625	0.0625	$0.615826833971767 \times 10^{-16}$	0.5625	0.0625	0
	0.1875	$0.676542155630955 \times 10^{-16}$		0.1875	0
	0.3125	$0.221975215985992 \times 10^{-12}$		0.3125	$0.774380559676047 \times 10^{-14}$
	0.4375	$0.321239562728337 \times 10^{-11}$		0.4375	$0.421190859967169 \times 10^{-12}$
	0.5625	$0.801352040280534 \times 10^{-12}$		0.5625	$0.959454737881060 \times 10^{-12}$
	0.6875	$0.625650226071528 \times 10^{-11}$		0.6875	$0.627997653879220 \times 10^{-12}$
	0.8125	$0.146282014279464 \times 10^{-10}$		0.8125	$0.437927472063393 \times 10^{-12}$
	0.9375	$0.27701820004553 \times 10^{-10}$		0.9375	$0.732747196252603 \times 10^{-13}$
0.1875	0.0625	$0.780625564189563 \times 10^{-16}$	0.6875	0.0625	0
	0.1875	$-0.624500451351651 \times 10^{-16}$		0.1875	0
	0.3125	$0.257953380877751 \times 10^{-12}$		0.3125	$0.879851747015437 \times 10^{-14}$
	0.4375	$0.401562116891796 \times 10^{-11}$		0.4375	$0.469235761357822 \times 10^{-12}$
	0.5625	$0.295388713489331 \times 10^{-12}$		0.5625	$0.131422650540003 \times 10^{-11}$
	0.6875	$0.829344926067677 \times 10^{-11}$		0.6875	$0.565436586441592 \times 10^{-12}$
	0.8125	$0.165019664599697 \times 10^{-10}$		0.8125	$0.288657986402541 \times 10^{-12}$
	0.9375	$0.291159596432777 \times 10^{-10}$		0.9375	$0.315303338993544 \times 10^{-12}$
0.3125	0.0625	$0.381639164714898 \times 10^{-16}$	0.8125	0.0625	0
	0.1875	$0.693889390390723 \times 10^{-17}$		0.1875	0
	0.3125	$0.918709552877317 \times 10^{-13}$		0.3125	$0.505151476204446 \times 10^{-14}$
	0.4375	$0.203401184784013 \times 10^{-11}$		0.4375	$0.150712775592865 \times 10^{-12}$
	0.5625	$0.191097138113605 \times 10^{-11}$		0.5625	$0.747513162480118 \times 10^{-12}$
	0.6875	$0.454117299319989 \times 10^{-11}$		0.6875	$0.772715225139109 \times 10^{-13}$
	0.8125	$0.574978953338245 \times 10^{-11}$		0.8125	$0.161870516990348 \times 10^{-12}$
	0.9375	$0.738953342960258 \times 10^{-11}$		0.9375	$0.394906329859168 \times 10^{-12}$
0.4375	0.0625	$0.346944695195361 \times 10^{-17}$	0.9375	0.0625	0
	0.1875	$0.138777878078145 \times 10^{-16}$		0.1875	0
	0.3125	$0.157374113740616 \times 10^{-13}$		0.3125	$0.888178419700125 \times 10^{-15}$
	0.4375	$0.362293528510804 \times 10^{-12}$		0.4375	$0.706656955173912 \times 10^{-13}$
	0.5625	$0.152178269985370 \times 10^{-11}$		0.5625	$0.960342916300760 \times 10^{-13}$
	0.6875	$0.923650045336899 \times 10^{-12}$		0.6875	$0.326849658449646 \times 10^{-12}$
	0.8125	$0.764777130513039 \times 10^{-12}$		0.8125	$0.272448730243013 \times 10^{-12}$
	0.9375	$0.349514861497369 \times 10^{-11}$		0.9375	$0.216160422894518 \times 10^{-12}$

4. Conclusion

In this paper, Chebyshev wavelet collocation method is applied to the Ginzburg Landau equation and also the numerical results are compared with exact solution of the equation. As shown from the figures and tables. It is clear that its numerical solution approaches the exact solution. Moreover, the implementation of this method is very suitable for solving initial and boundary value problems. In addition, the greatest advantage of this method is small computation costs and its simplicity.

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