POLYNOMIAL BASED DIFFERENTIAL QUADRATURE FOR NUMERICAL SOLUTIONS OF KURAMOTO-SIVASHINSKY EQUATION

***Gülsemay YİĞİT*[1] *and Mustafa BAYRAM*[*2]*,**

[1]School of Engineering and Natural Sciences, Altınbaş University, Istanbul, TURKEY

[*2]Department of Computer Engineering, Istanbul Gelişim University, Istanbul, TURKEY

[*] Corresponding author; E-mail: mbayram@gelisim.edu.tr

In this study, a numerical discrete derivative technique for solutions of Kuramoto-Sivashinksy equation is considered. According to the procedure, differential quadrature algorithm is adapted in space by using Chebyshev polynomials and explicit scheme is constructed to discretize time derivative. Sample problems are presented to support the idea. Numerical solutions are compared with exact solutions and also previous works. It is observed that the numerical solutions are well matched with the exact or existing solutions.

Key words: *Kuramoto-Sivashinski equation, The differential quadrature, Chebyshev polynomials.*

1. Introduction

Numerical techniques to solve partial differential equations which model many real life phenomenon, are widely used due to their fast and effective outcomes. Today, solving these kinds of problems both analytically or numerically attract many scientist. In this content Differential Quadrature method is considered to solve Kuramoto-Sivasinsky equation.

The Kuramato-Sivashinky equation has been presented as models of phase turbulence in reaction-diffusion systems [1, 2], plasma instabilities and flame front propagation [3]. The model equation has been widely studied analytically and numerically. Collocation methods based on Chebyshev spectral scheme [4], quintic B-splines [5], exponential cubic B-splines [6], have been considered.

Solutions of KS equation analyzed by using various methods such as finite difference methods [7], discontinuous Galerkin method [8], numeric meshless method for space derivatives using radial basis function [9], He's variational iteration method [10]. Rademacher and Wattenberg studied on viscous shocks for the model equation [11]. Also, some control results of the equation are presented [12, 13].

Differential Quadrature is discrete derivative method to solve ordinary or partial differential equations which gives numerical results effectively. The method presented by Bellman and his associates [14,15]. The idea was to give a new perspective for previous numerical techniques in solving problems. Since then, the way has been adapted in a wide range of applications. As to the the idea, the derivative of a function is defined as a weighted linear sum of the function values at all grid points related to the used direction. So, the term weighting coefficients occurs, and to obtain these coefficients, generally polynomials are chosen as test functions which can be obtained by polynomial approximation theory.

In the beginning, Bellman proposed the idea of two polynomial-based methods for computation of the weighting coefficients for the first order derivative. Power function was used as test function for the first approach. And the second one he choose test function as Legendre polynomials [14, 15]. Shortly after, many new different approaches have been presented with applications to many different kind of engineering problems. Popular ways have been suggested such as polynomials, Spline functions or Fourier series expansion. Quan and Chang [16, 17] used Legendre interpolation polynomials as test functions then obtained explicit formulation to find the weighting coefficients. Shu [18], gave a powerful way as a combination Bellman's and Quang-Chang's approaches. Also, first and higher order derivative formulations were analyzed in detail based on polynomial approaches and Fourier series expansion approaches using different kind of grid points. Stability analysis based on eigenvalue distribution was explained together with different time integration schemes. Shu also presented the relationships between finite difference and collocation methods with the differential quadrature method [18].

The DQ method has been effectively used in areas such as material science, thermal and structural mechanical analysis, physics and biology. And it can be seen that this technique gives accurate solutions with time saving computations [19]. Civan and Spliepcevich applied this method to both Poisson equation [20], and to multi-dimensional problems [21]. Saka et al. considered equal width equation (EW) by using three methods including cosine expansion based differential quadrature [22]. Alper et al, studied on a wide of range of problems using Spline functions or polynomials. For time discretization they used fourth order Runge-Kutta scheme and stability analysis is examined [23, 24, 25]. Sari and Güraslan investigated the polynomial based method for generalized Burgers-Huxley equation together third order third-order Runge-Kutta scheme for temporal discretization, without using linearization [26]. Mittal et al, used Bernstein polynomials to acquire the weighting coefficients [27]. In our work, we consider eigenvalue distrubition to check the stability, also, several theoretical works have been established for stability analysis of nonlinear partial differential equations [28,29].

The Kuramoto-Sivashinsky (KS) equation is a nonlinear partial differential equation given by,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^4 u}{\partial x^4}=0, \quad x\in[x_0,x_N], \quad t\in(0,T] \tag{1}$$

along with the boundary conditions

$$\begin{aligned} &u(x_0,t)=g_0, \quad u(x_N,t)=g_1, \\ &u_x(x_0,t)=0, \quad u_x(x_N,t)=0, \\ &u_{xx}(x_0,t)=0, \quad u_{xx}(x_N,t)=0, \end{aligned} \tag{2}$$

and initial condition

$$u(x,0)=u_0 \tag{3}$$

where α represents growth of the linear stability and ν shows surface tension. When $\nu=0$, the term surface tension is removed from the equation, then the equation becomes Burgers' equation [5]. In this work, it is used Chebyshev polynomial approximations to obtain numeric solutions. When the method is applied for the derivatives, differential equation is reduced to linear system of equation, with the implementation of boundary or initial conditions, matrix equation can be solved to obtain the desired solution.

2. The Differential Quadrature Method

Consider a sufficiently smooth function $f(x)$ on a closed interval $[a,b]$. Derivative of the function at a grid point x_i, is approximated by a linear sum of all functional values on the whole domain and the quadrature formula for first derivative is given as follows:

$$f_x(x_i)=\left.\frac{df}{dx}\right|_{x_i}=\sum_{j=1}^{N} w_{ij} f(x_j),\ i=1,2,\cdots,N \tag{4}$$

where w_{ij} represents the weighting coefficients to be evaluated, N is the number of grid points [18]. The n-th order derivative is defined as same idea given by,

$$f_x^{(n)}(x_i)=\left.\frac{df}{dx}\right|_{x_i}=\sum_{j=1}^{N} w_{ij}^{(n)} f(x_j),\ i=1,2,\cdots,N \tag{5}$$

where $w_{ij}^{(n)}$ represents the weighting coefficients, N is the number of grid points. The main idea according to the procedure is to determine weighting coefficients. The DQ method offers using uniform or nonuniform selection of grid points but, it gives more effective and stable solutions using Chebyshev-Gauss Labotto points [18, 19]. Here, we choose grid points as the Chebyshev collocation points defined as

$$x_i=\cos(\theta_i),\quad \theta=\frac{i\pi}{N},\quad i=1,2,\cdots,N \tag{6}$$

which is applicable for only interval [1,-1]. If the problem is given on interval $[a,b]$ to obtain x_i following transformation is used [18].

$$x_i=\frac{b-a}{2}(1-\xi_i)+a. \tag{7}$$

When Lagrange interpolating polynomials are considered as test function,

$$r_k(x)=\frac{K(x)}{(x-x_k)K'(x_k)},\quad i=1,2,\cdots,N \tag{8}$$

where $r_k(x)$ represents the test function and

$$K(x)=(x-x_1)(x-x_1)(x-x_2)\cdots(x-x_N),\quad K'(x)=\prod_{k=1,k\neq j}^{N}(x_j-x_k). \tag{9}$$

Using polynomial approach theory, for first order derivative the weighting coefficients, given in Eq. (4) becomes as follows [18]:

$$a_{ij}(x)=\frac{K'(x_i)}{(x_i-x_j)K'(x_j)},\quad j\neq i. \tag{10}$$

For the entries on main diagonal, the following relation becomes,

$$a_{ii}(x)=\frac{K''(x_i)}{(x_i-x_j)2K'(x_i)}. \tag{11}$$

By using the linear vector space spanning property as being represented by different kind of bases, on diagonal entries following formula can be used which is obtained by power functions, $x^{k-1}, \quad k=1,2,\cdots,N$ when $k=1$,

$$\sum_{j=1}^{N} a_{ij} = 0, \quad a_{ii} = -\sum_{j=1,\, j\neq i}^{N} a_{ij}. \tag{12}$$

Now, to obtain quadrature solutions of the model problem Chebyshev polynomial is used as a basis together with Chebyshev collocation points, the function $K(x)$ can be obtained as,

$$K(x) = (1-x^2)T_N^{(1)}(x) \tag{13}$$

where $T_N^{(1)}(x)$ represents first derivative of $T_N(x) = \cos(N\theta)$, and $\theta = \arccos(x)$. Thus,

$$T_N^{(1)}(x) = (\cos(N\theta))' = -N\sin(N\theta)d\theta \tag{14}$$

where

$$d\theta = -\frac{1}{\sqrt{1-x^2}} = -\frac{1}{\sin(\theta)} \tag{15}$$

The expression can be rewritten,

$$K(x) = K(\theta) = N\sin(\theta)\sin(N\theta). \tag{16}$$

Since, derivative approximation is needed according to the structure of the method, by differentiating Eq. (16),

$$K^{(1)}(x) = (N\sin(\theta)\sin(N\theta))' \tag{17}$$

or,

$$K^{(1)}(x) = -\frac{N\sin(N\theta)\cos(\theta) + N^2\sin(\theta)\cos(N\theta)}{\sin(\theta)}. \tag{18}$$

Since $N\theta_i = i\pi$, when $\sin(\theta_i) \neq 0$ that is $i \neq 0, N$, then Eq. (18) becomes

$$K^{(1)}(x_i) = (-1)^{i+1}N^2. \tag{19}$$

When $\sin(\theta_i) = 0$, to remove the undetermined form L'Hospital's rule is applied:

$$\begin{aligned} K^{(1)}(x_0) &= -2N^2, \\ K^{(1)}(x_N) &= (-1)^{N+1}2N^2. \end{aligned} \tag{20}$$

The reduced formulation related to find first derivative matrix interpreted as follows [18], [23]:

$$\begin{aligned} w_{ij} &= \frac{\bar{c}_i(-1)^{i+j}}{\bar{c}_j(x_i - x_j)}, \quad 0 \leq i, j \leq N, i \neq j, \\ w_{ii} &= -\frac{x_i}{2(1-x_i^2)}, \quad 1 \leq i \leq N-1, \\ w_{00} &= -w_{NN} = \frac{2N^2+1}{6} \end{aligned} \tag{21}$$

where $\overline{c_0} = \overline{c_N} = 2$ and $\overline{c_j} = 1, \quad 0 \le i \le N-1$. The model equation requires rewriting higher order derivatives in DQ formulations. In this manner, matrix multiplication method is used which mentioned in [18].

3. Quadrature Discretization of the Model Equation

The KS equation is rewritten as

$$U_t = -\ UU_x + \alpha U_{xx} + \nu U_{xxxx} \tag{22}$$

To obtain the approximated solution, we apply the method for each grid points, as follows,

$$U_t(x_i) = -\ U(x_i)U_x(x_i) + \alpha U_{xx}(x_i) + \nu U_{xxxx}(x_i) \tag{23}$$

Then, spatial derivatives are replaced by the Differential Quadrature equality,

$$\frac{\partial U(x_i)}{\partial t} = -\left(U(x_i)\sum_{j=1}^{N} w_{ij}^{(1)}U(x_j) + \alpha \sum_{j=1}^{N} w_{ij}^{(2)}U(x_j) + \nu \sum_{j=1}^{N} w_{ij}^{(4)}U(x_j) \right) \tag{24}$$

First order temporal discretization is obtained by forward Euler scheme,

$$U_t(x_i, t) = \frac{U^{n+1} - U^n}{\Delta t} \tag{25}$$

Matrix stability has been studied for the DQ discretized systems. For a discrete time-dependent problem is of the form,

$$\frac{\partial U}{\partial t} = \ell(U) \tag{26}$$

with proper initial and boundary conditions. Here, ℓ represents spatial nonlinear differential operator. After applying DQM and linearization of the nonlinear term $U(x)U_x(x)$ the equation becomes,

$$\frac{d\{U\}}{dt} = [A]\{U\} + \{g\} \tag{27}$$

where $\{U\}$ is an unknown vector of the function values in the domain, $\{g\}$ is the vector containing the nonhomogeneous part and the boundary conditions and A is the discretized coefficient matrix. The stability of the numerical discretized system depends on eigenvalues distribution [18]. The condition for absolute stability of the forward scheme is given by,

$$|1 + \lambda \Delta t| \le 1 \tag{28}$$

The stability region for the scheme is the circle with radius 1 and center (-1,0) on the complex $\lambda\Delta t$ plane [30]. When the the idea is implemented for the test problem 1, the maximum real parts of the eigenvalues are determined as 1.2225×10^{-11}, 6.6208×10^{-7}, 3.8338×10^{-8}, and 5.1920×10^{-4}, for $N = 10$, $N = 20$, $N = 30$, and $N = 60$ respectively. Eigenvalue distributions for each grid points are given by Fig. 1-4. The approximated solutions are obtained by combining the explicit scheme and and quadrature scheme by reducing the model to an algebraic system of equations. The solution of the matrix equation gives desired solution. Here, we used two sample problems to illustrate the efficiency of the presented method and all the results in terms of error norms are given in tables. In the end, solutions are also compared with previous studies. Solutions show approximately same accurateness.

4. Numerical Illustrations

Efficiency of the method is demonstrated using L_2 error norm which is given by,

$$L_2 = |U_{ex} - U_{nu}| = \left[\sum_{j=1}^{N} \left|(U_{ex})_j - (U_{nu})_j\right|^2\right]^{1/2} \quad i = 1, 2, \cdots, N \tag{29}$$

and maximum error norm given by

$$L_\infty = |U_{ex} - U_{nu}|_\infty = \max_j \left|(U_{ex})_j - (U_{nu})_j\right| \quad i = 1, 2, \cdots, N. \tag{30}$$

To compare the accuracy with previous studies it is also measured global relative error which is given by

$$GRE = \frac{\sum_{j=1}^{N} \left|(U_{ex})_j - (U_{nu})_j\right|}{\sum_{j=1}^{N} \left|(U_{ex})_j\right|} \quad i = 1, 2, \cdots, N \tag{31}$$

where U_{ex} and U_{nu} represents the analytical and numerical solutions, respectively.

Test Problem 1: As a first case, KS equation is considered for $\alpha = 1$ and $\nu = 1$. Exact solution of the problem is [5].

$$u(x,t) = b + \frac{15}{19}\sqrt{\frac{11}{19}}\left[-9\tanh(k(x - bt - x_0)) + 11\tanh^3(k(x - bt - x_0))\right]. \tag{32}$$

The initial condition can be computed by taking exact solution together with boundary conditions given by (1). Number of partitions are considered 15, 30, 60 and 150. Comparisons between the exact and numerical solutions are tabulated for the interval $[0,1]$. The solution models the shock wave propagation with speed b and initial position x_0. Solutions are given by Tab. 1.

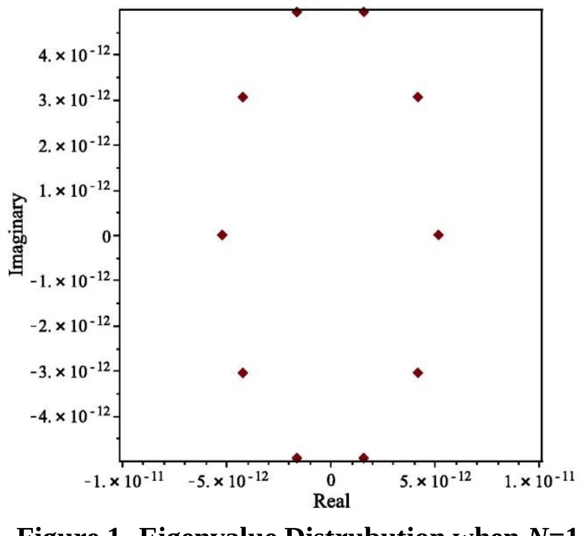


Figure 1- Eigenvalue Distrubution when *N*=10

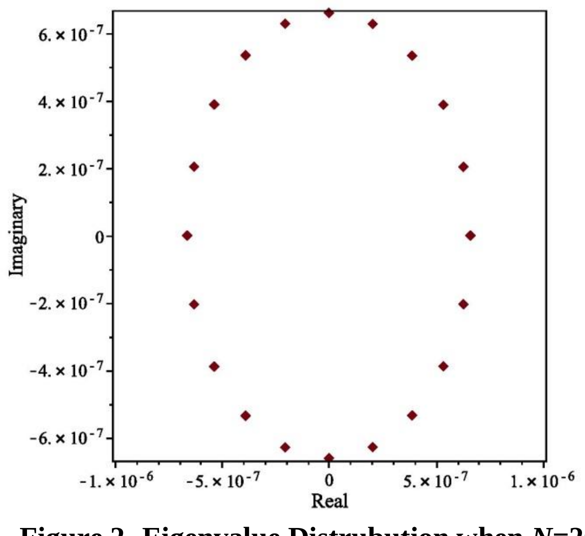
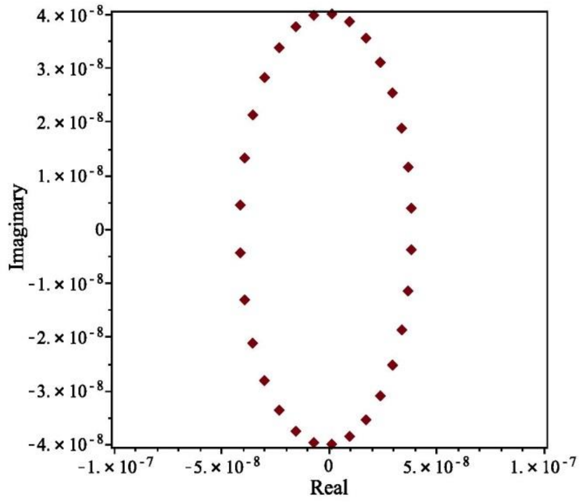


Figure 2- Eigenvalue Distrubution when *N*=20



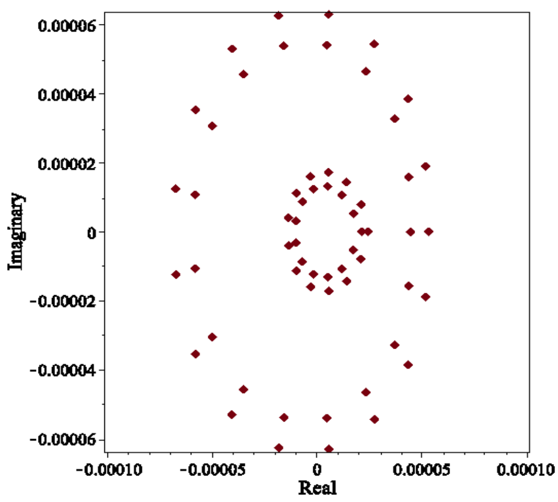


Figure 3- Eigenvalue Distrubution when *N*=30 **Figure 4- Eigenvalue Distrubution when *N*=60**

Table 1- Error Norms for Problem 1

Error Norms	N	$t=0.001$	$t=0.01$	$t=1.0$
L_2	15	3.5043E-05	4.4490E-03	9.7979E-03
	30	5.1571E-05	5.0774E-04	4.1705E-03
	60	1.4117E-04	7.4159E-04	1.7145E-02
	150	2.2567E-04	1.1758E-03	2.7408E-02
L_∞	15	1.3189E-05	1.1675E-03	3.6482E-03
	30	1.3286E-05	1.3080E-03	1.0742E-03
	60	2.2529E-04	1.3102E-04	3.0704E-03
	150	2.2529E-04	1.3108E-04	3.0718E-03

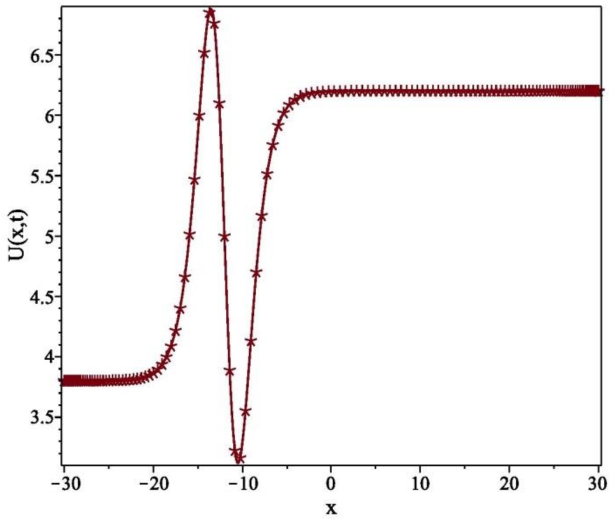


Figure 4- Comparision between numerical and exact solutions of KS equation when $N=150$ **Problem 1 (dotted line represents numerical solutions)**

Test Problem 2: Now, we consider the problem for $\alpha=-1$ and $\nu=1$. Exact solution of the problem is given by [5].

$$u(x,t)=b+\frac{15}{19}\sqrt{\frac{1}{19}}\left[-3\tanh(k(x-bt-x_0))+11\tanh^3(k(x-bt-x_0))\right] \qquad (33)$$

The initial condition is obtained by taking the exact solution together with boundary conditions given by. We have studied the algorithm with parameters $b=5$, $k=1/2\sqrt{19}$, $x_0=-25$. Number of partitions are considered 15 and 30 and 200. Comparisons between the exact and numerical solutions are tabulated for the interval $[-50,50]$.

Table 2- Error Norms for Problem 2

Error Norms	N	$t=0.001$	$t=0.01$	$t=1.0$
L_2	15	1.3980E-07	1.3823E-06	1.2138E-05
	30	2.0538E-07	2.0309E-06	1.7832E-05
	150	2.0538E-07	5.4047E-06	4.7457E-05
L_∞	15	4.7292E-08	4.6764E-07	4.1065E-06
	30	4.6969E-08	1.3080E-06	4.1240E-06
	150	4.7561E-08	4.7030E-07	4.1300E-06

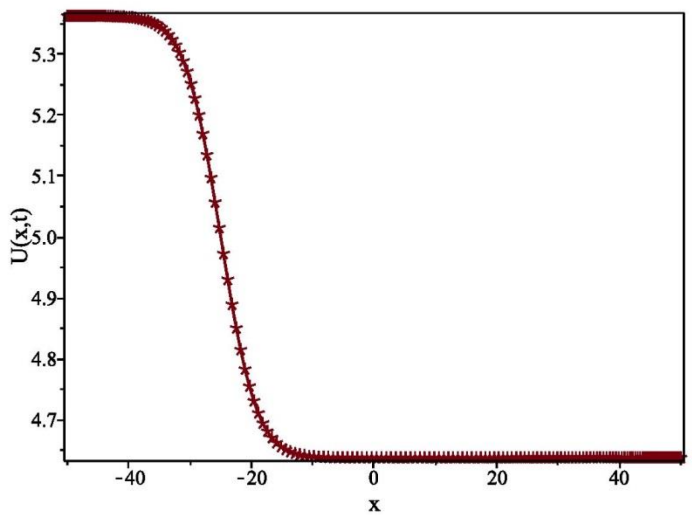


Figure 5- Comparision between numerical and exact solutions of KS equation when $N=200$ for Problem 2 (dotted line represents numerical solutions)

Table 3- Comparison of Global Relative Errors for Problem 1

Method	GRE
Chebyshev Differential Quadrature	1.1942E-03
Exponential B-Spline Collocation (Ersoy and Dag 2016) [6]	8.7463E-04
Quintic B-Spline Collocation (Mittal and Arora 2010) [5]	3.8172E-04
Lattice Boltzmann (Lai and Ma 2009) [31]	6.7923E-04

Table 4- Comparison of Global Relative Errors for Problem 2

Method	GRE
Chebyshev Differential Quadrature	1.6812E-05
Exponential B-Spline Collocation (Ersoy and Dag 2016) [6]	3.6467E-05
Quintic B-Spline Collocation (Mittal and Arora 2010) [5]	6.5092E-06
Lattice Boltzmann (Lai and Ma 2009) [31]	7.8088E-06

5. Conclusion

In this study, the Chebyshev based differential quadrature method is used for solutions of Kuramoto-Sivashinksy equation. The efficiency of the approach is examined by two examples. According to the Tab. 1 and Tab. 2 as t increases, accuracy decreases, but as the number of partitions increases, we can see approximately same accuracy. Also by Tab. 3 and Tab. 4, comparision with other methods shows that effectiveness is approximately same for similar numerical techniques. It can be also seen that from the figures numerical and exact solutions are in good agreement. The method is easy to implement by using small number of grid points.

References

[1] Kuramoto, Y., Tsuzuki, T., Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progress of theoretical physics*, *55* (1976), 2, pp.356-369

[2] Kuramoto, Y., and Tsuzuki T., On the Formation of Dissipative structures in reaction-diffusion systems, *Progress of theoretical physics, 54* (1975), 3, pp.687-699

[3] Sivashinsky, G.I., Instabilities, pattern-formation, and turbulence in flames, *Annual Review of Fluid Mechanics*, *15* (1983), 1, pp.179-199

[4] Khater, A.H., Temsah, R.S., Numerical solutions of the generalized Kuramoto-Sivashinsky equation by Chebyshev spectral collocation methods, *Computers and Mathematics with Applications, 56* (2008), 6, pp.1465-1472

[5] Mittal, R.C., Arora, G., Quintic B-spline collocation method for numerical solution of the Kuramoto-Sivashinsky equation, *Communications in Nonlinear Science and Numerical Simulation, 15* (2010), 10, pp.2798-2808

[6] Ersoy, O., Dag, I., The exponential cubic B-Spline collocation method for the Kuramoto-Sivashinsky equation, *Filomat, 30* (2016), 3, pp.853-861

[7] Akrivis, G.D., Finite difference discretization of Kuramoto-Sivashinsky equation, *Numerische Mathematik, 63* (1992), 1, pp.1-11

[8] Xu, Y., Shu C.W., Local discontinuous Galerkin methods for the Kuramoto-Sivashinsky equations and the Ito-type coupled KdV equations, *Computer methods in applied mechanics and engineering, 195* (2006), 25-28, pp.3430-3447

[9] Uddin, M., Haq, S., Islam, S., A mesh-free numerical method for solution of the family of Kuramoto-Sivashinsky equations, *Applied Mathematics and Computation, 212* (2009), 2, pp.458-69

[10] Porshokouhi, M.G., Ghanbari, B., Application of He's variational iteration method for solution of the family of Kuramoto-Sivashinsky equations, *Journal of King Saud University-Science, 23* (2011), 4, pp.407-411

[11] Rademacher, J., Wattenberg, R., Viscous shocks in the destabilized Kuramoto-Sivashinsky equation, *Journal of computational and nonlinear dynamics, 1* (2006), 4, pp.336-347

[12] Cerpa, E., Guzman P., Mercado A., On the control of the linear Kuramoto-Sivashinsky equation, *ESAIM: Control, Optimisation and Calculus of Variations, 23* (2017), 1, pp.165-194

[13] Cerpa, E., Nicolas C., Local controllability of the stabilized Kuramoto-Sivashinsky system by a single control acting on the heat equation, *Journal de Mathématiques Pures et Appliquées, 106* (2016), 4, pp.670-694

[14] Bellman, R.E., Casti J., Differential quadrature and long term integration *Journal of Mathematical Analysis and Applications, 34* (1971), 2, pp.235-238

[15] Bellman, R.E., Kashef B.G., Casti J., Differential Quadrature: A Technique for the rapid solution of nonlinear partial differential equations, *Journal of Computer Physics, 10* (1972), 1, pp.40-52

[16] Quan, J., Chang C., New Insights in solving distributed system equations by the quadrature method-I. Analysis, *Computers and Chemical Engineering, 13* (1989), 7, pp.779-788

[17] Quan, J., Chang C., New Insights in solving distributed system equations by the quadrature method-II. Analysis, *Computers and Chemical Engineering, 13* (1989), 9, pp.1017-1024

[18] Shu, C., *Differential quadrature and its applications in engineering*, Springer, Great Britain, (2000)

[19] Bert, C.W., Malik, M., Differential quadrature method in computational mechanics: A Review, *Applied mechanics reviews, 49* (1996), 1, pp.1-28

[20] Civan, F., Sliepcevich, C.M., Solution of the Poisson equation by differential quadrature *International Journal for Numerical Methods in Engineering, 19* (1983), 5, pp.711-724

[21] Civan, F., Sliepcevich, C.M., Differential quadrature for multidimensional problems, *Journal of Mathematical Analysis and Applications, 101* (1983), pp.423-443

[22] Saka, B., Dağ, I., Dereli, Y., Korkmaz, A., Three Different Methods for Numerical Solution of the EW Equation, *Engineering Analysis with Boundary Elements, 32* (2008), 7, pp.556-566

[23] Korkmaz, A., Dağ, I., Polynomial based differential quadrature method for numerical solutions of nonlinear Burgers equation, *Journal of the Franklin Institute, 348* (2011), 10, pp.2863-2875

[24] Korkmaz, A., Dağ, I., Quantic and quintic B-Spline methods for advection-diffusion Equation, *Applied Mathematics and Computation, 274* (2016) pp.208-219

[25] Korkmaz, A., Dağ, I., Quantic and quintic B-Spline methods for advection-diffusion Equation, *Applied Mathematics and Computation, 274* (2016) pp.208-219

[26] Sari, M., Gürarslan, G., Numerical solutions of the generalized Burgers-Huxley equation by a differential quadrature method, *Mathematical Problems in Engineering, 2009* (2009)

[27] Mittal, R.C., Arora, G., A study of one dimensional nonlinear diffusion equations by Bernstein polynomial based differential quadrature, *Journal of Mathematical Chemistry, 55* (2017), 2, pp.673-695

[28] Inc, M., Yusuf, A., Aliyu, A. I., Baleanu, D., Soliton solutions and stability analysis for some conformable nonlinear partial differential equations in mathematical physics, *Optical and Quantum Electronics, 50*, (2018), 190

[29] Baleanu, D., Inc, M., Yusuf, A., and Aliyu, A. I., Traveling wave solutions and conservation laws for nonlinear evolution equation. *Journal of Mathematical Physics*, *59,* (2018), 2, 023506

[30] Kumar, B. R., Mehra M., Wavelet-Taylor Galerkin method for the Burgers equation, *BIT Numerical Mathematics, 45,* (2005), 3, pp.543-560

[31] Lai, H., and Ma, C., Lattice Boltzmann method for the generalized Kuramoto–Sivashinsky equation. *Physica A: Statistical Mechanics and its Applications*, *388,* (2009), 8, pp.1405-1412