DISCRETE FRACTIONAL SOLUTION OF A NONHOMOGENEOUS NON-FUCHSIAN DIFFERENTIAL EQUATIONS

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In this article, we also present new fractional solutions of the non-homogeneous and homogeneous non-Fuchsian differential equation by using nabla-discrete fractional calculus operator $\nabla^{\alpha}(0<\alpha<1)$. *So, we acquire new solution of these equation in the discrete fractional form via a newly developed method.*

Key words*: Discrete fractional calculus; nabla operator; non-Fuchsian equations.*

Introduction

Discrete fractional calculus deals with sums and differences of arbitrary orders. Fractional calculus has many applications in diverse fields of science and engineering; such as, viscoelasticity, diffusion, heat transfer, overall thermal conductivity, neurology, control theory, and statistics [1-5]. A similar theory was started for discrete fractional calculus and the definition and properties of fractional sums and differences theory were developed. Many article related to this topic have seemed lately [6-20]. Finding exact solutions to nonlinear partial differential equations defining the evolution of localized waveforms is a important subject in nonlinear science [21–23].

In 1956 [6], differences of fractional order was first introduced by Kuttner.

Diaz and Osler [7], defined concept of the fractional difference as follows

$$\Delta^{\vartheta}\varphi(\tau)=\sum_{k=0}^{\infty}(-1)^{k}\binom{\vartheta}{k}\varphi(\tau+\vartheta-k)$$

where ϑ is any real number.

Granger and Joyeux [24] and Hosking [25], defined concept of the fractional difference as follows

$$\begin{aligned}\nabla^{\vartheta}\varphi(\tau)&=(1-q)^{\vartheta}\varphi(\tau)\\&=\sum_{k=0}^{\infty}(-1)^{k}\frac{\Gamma(\vartheta+1)}{\Gamma(k+1)\Gamma(\vartheta-k+1)}q^{k}\varphi(\tau)\\&=\sum_{k=0}^{\infty}(-1)^{k}\binom{\vartheta}{k}\varphi(\tau-k),\end{aligned}$$

where ϑ is real number and $q\varphi(\tau)=\varphi(\tau-1)$ is the standart shift operator. The notation ∇^{ϑ} is used since this definition is a natural extension of the backward difference operator.

The theory for fractional nabla difference calculus was initiated by Gray and Zhang [26], Atici and Eloe [10], Acar and Atıcı [12], and Anastassiou [27], where basic approaches, definitions and properties of fractional sums and differences were reported.

In this article, we will obtain explicit solutions for a general class of second order ordinary differential equations. For this, we will take advantage of from the definition, theorems and properties of discrete fractional analysis.

Preliminary and Properties

Let $\vartheta \in \mathbb{R}^{+}$ such that $n-1 \le \vartheta < n$ where n is an integer. $\vartheta - th$ order fractional sum of φ is given by

$$\nabla_a^{-\vartheta}\varphi(t) = \frac{1}{\Gamma(\vartheta)}\sum_{\tau=a}^{t}\left(t-\sigma(\tau)\right)^{\overline{\vartheta-1}}\varphi(\tau), \tag{1}$$

where $t \in \mathbb{N}_a = \{a, a+1, a+2, ...\}$, a $\in \mathbb{R}$, $\sigma(t) = t-1$ is the jump operator on the time scala calculus.

The ascending factorial is defined by

$$t^{\bar{n}} = t(t+1)(t+2)...(t+n-1), n \in \mathbb{N}, t^{\bar{0}} = 1.$$

Let $\vartheta \in \mathbb{R}$. Then '' t to the ϑ raising'' is given by

$$t^{\bar{\vartheta}} = \frac{\Gamma(t+\vartheta)}{\Gamma(t)}, \mathrm{t} \in \mathbb{R} \setminus \{..., -2, -1, 0\}, 0^{\bar{\vartheta}} = 0. \tag{2}$$

Note that

$$\nabla\left(t^{\bar{\vartheta}}\right) = \vartheta t^{\overline{\vartheta-1}}, \tag{3}$$

where $\nabla w(t) = w(t) - w(\sigma(t)) = w(t) - w(t-1)$.

$\vartheta - th$ order fractional difference of φ is defined by

$$\begin{aligned}\nabla_a^{\vartheta}\varphi(t) &= \nabla^{n}\left[\nabla^{-(n-\vartheta)}\varphi(t)\right] \\ &= \nabla^{n}\left[\frac{1}{\Gamma(n-\vartheta)}\sum_{\tau=a}^{t}\left(t-\sigma(\tau)\right)^{\overline{n-\vartheta-1}}\varphi(\tau)\right],\end{aligned} \tag{4}$$

where φ is defined on $\mathbb{N}_a = \{a, a+1, a+2, ...\}$ [12].

Theorem 1 [18]. Let φ and $\psi : \mathbb{N}_0^{+} \to \mathbb{R}, \delta, \eta > 0$ and h, k are scalas. The following equalities holds:

$$\nabla^{\delta}\left[h\varphi(t) + k\psi(t)\right] = h\nabla^{\delta}\varphi(t) + k\nabla^{\delta}\psi(t), \tag{5}$$

$$\nabla\nabla^{-\delta}\varphi(t) = \nabla^{-(\delta-1)}\varphi(t), \tag{6}$$

$$\nabla^{-\delta}\nabla\varphi(t) = \nabla^{(1-\delta)}\varphi(t) - \binom{t+\delta-2}{t-1}\varphi(0). \tag{7}$$

Lemma 2 (Power Rule) [19]. Let $v > 0$ and $\vartheta > -1$. Then

$$\nabla_a^{-v}(t-a)^{\bar{\vartheta}} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+v+1)}(t-a)^{\overline{\vartheta+v}}, \quad t \in \mathbb{N}_a, \tag{8}$$

$$\nabla_a^{\nu}(t-a)^{\overline{\vartheta}}=\frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta-\nu+1)}(t-a)^{\overline{\vartheta-\nu}},\quad t\in\mathbb{N}_{a+N}. \tag{9}$$

Lemma 3 (Leibniz Rule) [19]. Suppose that $\varphi(t)$ and $\psi(t)$ are analytic and single-valued functions. Then

$$\nabla_0^{\vartheta}(\varphi\psi)(t)=\sum_{n=0}^{t}\binom{\vartheta}{n}\left[\nabla_0^{\vartheta-n}\varphi(t-n)\right]\left[\nabla^{n}\psi(t)\right],\ \vartheta\in\mathbb{R};t\in\mathbb{C} \tag{10}$$

$$\binom{\vartheta}{n}=\frac{\Gamma(\vartheta+1)}{\Gamma(n+1)\Gamma(\vartheta-\nu+1)}$$

where $\nabla^{n}\psi(t)=\psi_n(t)$ is the ordinary derivative of $\psi(t)$ of order $n\in\mathbb{N}_0$, it being assumed that $\psi(t)$ is the polynomial part of the product $\varphi\psi$.

Definition 4. λ shift operator is defined by

$$\lambda^{n}\varphi(t)=\varphi(t-n) \tag{11}$$

where $n\in\mathbb{N}$ [28].

Lemma 5 (Index law) [20]. Suppose that $\varphi(t)$ is an analytic and single-valued function. Then

$$\left(\varphi_{\delta}(t)\right)_{\eta}=\varphi_{\delta+\eta}(t)=\left(\varphi_{\eta}(t)\right)_{\delta}\ \left(\varphi_{\delta}(t)\neq 0;\varphi_{\eta}(t)\neq 0;\delta,\eta\in\mathbb{R};t\in\mathbb{C}\right). \tag{12}$$

Main Results

In this section, we give two theorems for solutions of the nonhomogeneous and homogeneous linear differential equation by using the discrete fractional calculus operator.

Theorem 6. If the given functions u and g satisfies the constrain

$$u\in\left\{u:0\neq\left|u_{\alpha}(r)\right|<\infty,\alpha\in\mathbb{R}\right\}$$

and

$$g\in\left\{g:0\neq\left|g_{\alpha}(r)\right|<\infty,\alpha\in\mathbb{R}\right\}$$

then, the following nonhomogeneous linear differential equation

$$u_2Pr^2+u_1Qr+\left(Rr^2+Sr+T\right)u(r)=g(r)\quad\left(P\neq 0,R\neq 0\right) \tag{13}$$

has a particular solutions in the form;

$$u(r)=r^{\tau}e^{\kappa r}\left\{r^{-\left(\lambda\alpha+2\tau+\frac{Q}{P}\right)}e^{-2\kappa r}\left[P^{-1}\left(r^{-\tau-1}e^{-\kappa r}g\right)_{\alpha}r^{\lambda\alpha+2\tau+\frac{Q}{P}-1}e^{2\kappa r}\right]_{-1}\right\}_{-(1+\alpha)}, \tag{14}$$

where $u_n=\dfrac{d^nu}{dr^n}\ (n=0,1,2),\ u_0=u=u(r),\ \kappa,\tau$ are given by

$$\kappa = \pm i\sqrt{\frac{R}{P}}, \quad \tau = \frac{P-Q\pm\sqrt{(P-Q)^2-4PT}}{2P} \tag{15}$$

and

$$\alpha = -\lambda^{-1}\frac{(2P\tau+Q)\kappa+S}{2P\kappa} \tag{16}$$

Proof. Let

$$u(r) = r^{\tau}e^{\kappa r}w(r) \tag{17}$$

so that

$$\frac{du}{dr} = r^{\tau-1}e^{\kappa r}\left[r\frac{dw}{dr}+(\tau+\kappa r)w(r)\right] \tag{18}$$

and

$$\frac{d^2u}{dr^2} = r^{\tau-2}e^{\kappa r}\left[r^2\frac{d^2w}{dr^2}+2(\tau+\kappa r)r\frac{dw}{dr}+\left\{\kappa^2r^2+2\tau\kappa r+\tau(\tau-1)\right\}w(r)\right]. \tag{19}$$

Upon substituting from (17)-(19) into the non-Fuchsian linear ordinary differential equation (13), we easily obtain the transformed equation

$$\begin{aligned} &w_2Pr+w_1\left[2P(\tau+\kappa r)+Q\right] \\ &\quad+\left[(P\kappa^2+R)r+(2P\tau\kappa+Q\kappa+S)+\left\{P\tau(\tau-1)+Q\tau+T\right\}r^{-1}\right]w(r) \\ &\quad\quad= r^{-\tau-1}e^{-\kappa r}g(r). \end{aligned} \tag{20}$$

Finally, we find it to be suitable to restrict the diverse parameters related in (13) and (20) by means of the following equations

$$P\kappa^2+R=0, \quad \kappa=\pm i\sqrt{\frac{R}{P}}, \tag{21}$$

$$P\tau(\tau-1)+Q\tau+T, \quad \tau=\frac{P-Q\pm\sqrt{(P-Q)^2-4PT}}{2P}. \tag{22}$$

Under the parametric constraints given by (21) and (22) the non-Fuchsian differential equation (20) would reduce directly to the relatively simpler form

$$w_2Pr+w_1(2P\tau+Q+2P\kappa r)+w(r)\left[(2P\tau+Q)\kappa+S\right]=r^{-\tau-1}e^{-\kappa r}g(r), \tag{23}$$

where κ and τ are given by (21) and (22).

Operate ∇^{α} to the both sides of (23), we then obtain

$$\begin{aligned} &\nabla^{\alpha}(w_2Pr)+\nabla^{\alpha}\left[w_1(2P\tau+Q+2P\kappa r)\right]+\nabla^{\alpha}\left\{w\left[(2P\tau+Q)\kappa+S\right]\right\} \\ &\quad= \nabla^{\alpha}\left(r^{-\tau-1}e^{-\kappa r}g(r)\right). \end{aligned} \tag{24}$$

Using (10) and (12) we have

$$\nabla^{\alpha}\left(w_2 Pr\right) = w_{2+\alpha} Pr + P\lambda\alpha w_{1+\alpha} \tag{25}$$

and

$$\nabla^{\alpha}\left[w_1\left(2P\tau + Q + 2P\kappa r\right)\right] = w_{1+\alpha}\left(2P\tau + Q + 2P\kappa r\right) + 2P\lambda\alpha\kappa w_{\alpha}, \tag{26}$$

where λ is a shift operator. Making use of the relations (25) and (26), we may write (24) in the following form,

$$\begin{aligned} &w_{2+\alpha} Pr + w_{1+\alpha}\left(P\lambda\alpha + 2P\tau + Q + 2P\kappa r\right) \\ &\quad + w_{\alpha}\left[2P\lambda\alpha\kappa + \left(2P\tau + Q\right)\kappa + S\right] = \left(r^{-\tau-1} e^{-\kappa r} g\right)_{\alpha}. \end{aligned} \tag{27}$$

Choose α such that

$$2P\lambda\alpha\kappa + \left(2P\tau + Q\right)\kappa + S = 0,$$

$$\alpha = -\lambda^{-1}\frac{\left(2P\tau + Q\right)\kappa + S}{2P\kappa}$$

we have then

$$w_{2+\alpha} Pr + w_{1+\alpha}\left(P\lambda\alpha + 2P\tau + Q + 2P\kappa r\right) = \left(r^{-\tau-1} e^{-\kappa r} g\right)_{\alpha} \tag{28}$$

from (27).

Next, writing

$$w_{1+\alpha} = v\left(r\right) \quad \left(w = v_{-(1+\alpha)}\right), \tag{29}$$

we obtain

$$v_1 + v\left[\frac{\lambda\alpha + 2\tau + Q/P}{r} + 2\kappa\right] = \left(r^{-\tau-1} e^{-\kappa r} g\right)_{\alpha}\left(Pr\right)^{-1} \tag{30}$$

from (28) and (29). This is an ordinary differential equation which has a particular solution,

$$v = r^{-\left(\lambda\alpha + 2\tau + \frac{Q}{P}\right)} e^{-2\kappa r}\left[P^{-1}\left(r^{-\tau-1} e^{-\kappa r} g\right)_{\alpha} r^{\lambda\alpha + 2\tau + \frac{Q}{P} - 1} e^{2\kappa r}\right]_{-1}. \tag{31}$$

Therefore, we have (14) from (17),(29), and (31).

Theorem 7. Let $u \in \left\{u : 0 \neq \left|u_{\alpha}\left(r\right)\right| < \infty, \alpha \in \mathbb{R}\right\}$. Then, the following homogeneous linear differential equation

$$u_2 Pr^2 + u_1 Qr + \left(Rr^2 + Sr + T\right)u\left(r\right) = 0 \quad \left(P \neq 0, R \neq 0\right), \tag{32}$$

has a particular solution in the form;

$$u\left(r\right) = hr^{\tau} e^{\kappa r}\left\{r^{-\left(\lambda\alpha + 2\tau + \frac{Q}{P}\right)} e^{-2\kappa r}\right\}_{-(1+\alpha)}, \tag{33}$$

where $u_n = \dfrac{d^n u}{dr^n}\left(n = 0,1,2\right)$, $u_0 = u = u\left(r\right)$, h is an arbitrary constant, κ and τ are given by

$$\kappa = \pm i\sqrt{\frac{R}{P}}, \quad \tau = \frac{P-Q \pm \sqrt{(P-Q)^2 - 4PT}}{2P} \tag{34}$$

and

$$\alpha = -\lambda^{-1}\frac{(2P\tau + Q)\kappa + S}{2P\kappa}. \tag{35}$$

Proof. When $g = 0$ in Theorem 6, we conclude that

$$v_1 + v\left[\frac{\lambda\alpha + 2\tau + Q/P}{r} + 2\kappa\right] = 0 \tag{36}$$

instead of Equation (30). As a results, we obtain (33) for (36).

Conclusions

In this work, we applied the nabla operator to the non-homogeneous non-Fuchsian differential equation. We obtained the discrete fractional solutions of these equations via this new operator method. In our future works, we will obtain a fractional solution of similar type equations by using the Nabla operator of discrete fractional analysis.

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