A SPECIFIC STURM-LIOUVILLE DIFFERENTIAL EQUATION

Tanfer TANRIVERDI

Harran University Faculty of Arts and Sciences

Department of Mathematics Şanlıurfa Turkey 63290

E-mail: ttanriverdi@harran.edu.tr

Explicit solutions of the second order ordinary differential equation

$$y''(x)+(\lambda+12\,sech^2x)y(x)=0$$

and its eigenpairs are obtained by calculating complex residues. Eigenpairs (energey states and energy function), spectral function and eigenfunction expansions are also reported.

Key words: Sturm-Liouville, Schrödinger equation, energy equation, energy states, eigenfunctions, complex residues, explicit solution

1. Introduction

In this paper, we consider the following differential differential equation

$$y''(x)+\big(\lambda-q(x)\big)y(x)=0, \tag{1}$$

with specific potential function $q(x)=-12\text{sech}^2x$. Theory related to (1) is fully given in [1]. One may say that the only case to solve (1) explicitly for all λ is the case $q(x)=0$ where the solutions are oscillating. In fact, this is not correct. There are indeed some examples which are not reported yet. One such example is

$$y''(x)+(\lambda+12\text{sech}^2x\,)y(x)=0. \tag{2}$$

This example is slightly different from what Titchmarsh does for

$$y''(x)+\left(\lambda-\tfrac{1}{4}\sec^2x\right)y(x)=0,\ x\in\left(-\tfrac{\pi}{2},\tfrac{\pi}{2}\right),$$

whose solutions involve Legendre functions. For similar problems and results, see [1,2,3,4,5,6,7,8].

Schrödinger equation predicting the future behavior of a dynamic system is used in physics describing quantum mechanical behavior such as thermal conduction, most of chemistry dealing with problems about the atomic structure of matter and wave mechanics giving us information about the particle's behavior in time and space, etc. [9,10,11,12].

It is important to mention that why we study the problem in the subject line above. The solutions are oscillating when the energy of the particle is greater than the energy of the potential. Those regions are regions where a classical particle can exist.

2. Preliminaries

To get the expansion of an arbitrary function $f(x)$ in terms of eigenfunctions one finally needs to know the following definitions and lemmas taken from [1,3]. If $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are the solutions of (1), with α is real, and satisfying

$$\phi(0,\lambda) = \sin\alpha, \quad \phi'(0,\lambda) = -\cos\alpha, \quad \theta(0,\lambda) = \cos\alpha, \quad \theta'(0,\lambda) = \sin\alpha, \tag{3}$$

then Wronskian $W_x(\theta,\phi) = \cos^2\alpha + \sin^2\alpha = 1$. It is also well-known that the general solution of (1) is

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \in L^2(0,\infty).$$

The definiton of the non-decreasing spectrum function, for all real λ, is given by

$$k(\lambda) = \lim_{\delta\to 0} \int_0^\lambda -\mathrm{Im}\{m(u+i\delta)\}du\,.$$

Lemma 1 For suitable class of functions on $(0,\infty)$, one has

$$f(x) = \frac{1}{\pi}\int_0^\infty \phi(x,\lambda)dk(\lambda)\int_0^\infty \phi(t,\lambda)f(t)\,dt.$$

If $m(\lambda) = 0$ has poles, then $f(x)$ has the following extra terms

$$\sum_{N=0}^\infty \phi_N(x,\lambda)\int_0^\infty \phi_N(t,\lambda)f(t)\,dt. \tag{4}$$

Lemma 2 Let $q(x)$ be given. (If $q(x)$ is an even function, then $m_1(\lambda) = -m_2(\lambda)$) Then for a suitable class of functions on $(-\infty,\infty)$ one has the following expansion

$$f(x) = \frac{1}{\pi}\int_{-\infty}^\infty \theta(x,\lambda)d\xi\int_{-\infty}^\infty \theta(y,\lambda)f(y)\,dy + \frac{1}{\pi}\int_{-\infty}^\infty \phi(x,\lambda)d\zeta\int_{-\infty}^\infty \phi(y,\lambda)f(y)\,dy$$

where

$$\psi_1(x,\lambda) = \theta(x,\lambda) + m_1(\lambda)\phi(x,\lambda) \in L^2(-\infty,0),$$
$$\psi_2(x,\lambda) = \theta(x,\lambda) + m_2(\lambda)\phi(x,\lambda) \in L^2(0,\infty),$$

$$\begin{cases} \xi(\lambda) = \lim\limits_{\delta\to 0}\int_0^\lambda -\mathrm{Im}\left\{\frac{1}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du, \\ \xi'(\lambda) = \mathrm{Im}\left\{\frac{1}{2m_2(\lambda)}\right\} \text{ if } q(x) \text{ is an even function,} \end{cases} \tag{5}$$

$$\begin{cases} \zeta(\lambda) = \lim\limits_{\delta\to 0}\int_0^\lambda -\mathrm{Im}\left\{\frac{m_1(u+i\delta)m_2(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du\,, \\ \zeta'(\lambda) = \mathrm{Im}\left\{\frac{-1}{2m_2(\lambda)}\right\} \text{ if } q(x) \text{ is an even function.} \end{cases} \tag{6}$$

If $m(\lambda) = 0$ has poles, then the expansion of $f(x)$ has extra terms as in (4).

3. Main Results

We deal with the solution of (2) which is given by, with $s^2 = -\lambda$,

$$y(x) = \cosh^4 x\int_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^4}dz, \tag{7}$$

where simple closed curve C is only including the point $z = x$ and exluding the other zeros of $\sinh z - \sinh x$.

Theorem 1 The equation (7) solves the equation (2).

Proof.

$$y'(x) = 4\cosh^3 x\sinh x\oint\frac{\cosh(zs)}{(\sinh z-\sinh x)^4}dz + 4\cosh^5 x\oint\frac{\cosh(zs)}{(\sinh z-\sinh x)^5}dz,$$
$$= 4\tanh x\, y(x) + 4\cosh^5 x\oint\frac{\cosh(zs)}{(\sinh z-\sinh x)^5}dz.$$

$$y'' = 4\,\mathrm{sech}^2(x)\,y + 16\tanh x\left(\tanh(x)\,y + \cosh^5 x\oint_C \frac{\cosh(zs)}{(\sinh z-\sinh x)^5}dz\right)$$
$$+20\cosh^4 x\sinh x\oint_C \frac{\cosh(zs)}{(\sinh z-\sinh x)^5}dz + 20\cosh^6 x\oint_C \frac{\cosh(zs)}{(\sinh z-\sinh x)^6}dz,$$

$$y''(x) = 16y(x) - 12\text{sech}^2(x\,)y(x) + 36\cosh^4 x \sinh x \oint_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^5} dz$$
$$+ 20\cosh^6 x \oint_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^6} dz.$$

Thus,

$$y'' + 12\,\text{sech}^2(x)\,y(x)$$
$$= 4\cosh^4 x \oint_C \frac{\cosh(zs)\,(4(\sinh\ z - \sinh x)^2 + 9\sinh\ x\,(\sinh z - \sinh x) + 5\cosh^2 x)}{(\sinh z - \sinh x)^6},$$
$$y'' + 12\text{sech}^2(x)\text{y} = 4\cosh^4 x \oint_C \frac{\cosh(zs)(4\sinh^2 \text{z} + \sinh z \sinh x + 5)}{(\sinh z - \sinh x)^6} dz. \tag{8}$$

But also, employing integration by parts twice,

$$y(x) = \cosh^4 x \oint_C \frac{\cosh(zs)}{(\sinh z - \sinh x)^4} dz$$

$$= \frac{4\cosh^4 x}{s} \oint_C \frac{\sinh(zs)\cosh z}{(\sinh z - \sinh x)^5} dz$$
$$= \frac{4\cosh^4 x}{s^2} \oint_C \frac{\cosh(zs)(5\cosh^2 z - \sinh z(\sinh z - \sinh x))}{(\sinh z - \sinh x)^6} dz$$
$$= \frac{4\cosh^4 x}{s^2} \oint_C \frac{\cosh(zs)(4\sinh^2 z + \sinh z \sinh x + 5)}{(\sinh z - \sinh x)^6} dz,$$

by using $s^2 = -\lambda$,

$$\lambda y(x) = -4\cosh^4 x \oint_C \frac{\cosh(zs)(4\sinh^2 z + \sinh z \sinh x + 5)}{(\sinh z - \sinh x)^6} dz. \tag{9}$$

Comparing (8) and (9), we see that

$$y''(x) + (\lambda + 12\text{sech}^2 x)y(x) = 0.$$

Corollary 1 The factor $\cosh(zs)$ in (7) plays little part in the argument. By replacing $\cosh(zs)$ by $\sinh(zs)$. Hence, the other solution of (2) is given by, with $s^2 = -\lambda$,

$$y(x) = \cosh^4 x \oint_C \frac{\sinh(zs)}{(\sinh z - \sinh x)^4} dz. \tag{10}$$

Theorem 2 The equation (10) solves the equation (2).

Proof. The proof is the same as Theorem 1. Hence, it is excluded.

4. Explicit solutions and eigenvalues obtained by evaluating residues

We now require the residues of (7) and (10). The Taylor expansion around the point $z = x$ for numerator and denominator separately, one gets

$$\frac{\cosh(zs)}{(\sinh z - \sinh x)^4}$$
$$= \frac{\cosh(xs) + s(z-x)\sinh(xs) + \frac{(z-x)^2 s^2}{2}\cosh(sx) + \cdots}{(z-x)^4 \cosh^4(x)\left\{1 + \left(\frac{z-x}{2} + \frac{(z-x)^3}{4!} + \frac{(z-x)^5}{6!} + \cdots\right)\tanh x + \left(\frac{(z-x)^2}{3!} + \frac{(z-x)^4}{5!} + \cdots\right)\right\}^4}.$$

Getting the coefficient of $(z-x)^{-1}$ in the above expansion, so the residue at $z = x$ is

$$\frac{\cosh(xs)\,\{(9 - 6\text{s}^2)\tanh x - 15\tanh^3 x\} + s\sinh(xs)\{-4 + 15\tanh^2 x + s^2\}}{3!\cosh^4 x}.$$

By using $s^2 = -\lambda$, so that one solution is

$$y_1(x) = -\sqrt{\lambda}\sin(x\sqrt{\lambda})\{-4 + 15\tanh^2 x - \lambda\} + \cos(x\sqrt{\lambda})\{(9 + 6\lambda)\tanh x - 15\tanh^3 x\}.$$

In the same manner, one obtains the residue of (10) as

$$\frac{\sinh(xs)\{(9 - 6s^2)\tanh x - 15\tanh^3 x\} + s\cosh(xs)\{-4 + 15\tanh^2 x + s^2\}}{3!\cosh^4 x}.$$

Hence, another solution is similarly obtained as

$$y_2(x) = \sqrt{\lambda}\cos(x\sqrt{\lambda})\{-4 + 15\tanh^2 x - \lambda\} + \sin(x\sqrt{\lambda})\{(9 + 6\lambda)\tanh x - 15\tanh^3 x.\}$$

From (7) and (10), set $Y(x) = (7) - (10)$. Then

$$Y(x,s) = \cosh^4 x \oint_C \frac{e^{-sz}}{(\sinh z - \sinh x)^4}\,dz.$$

Theorem 3

$$Y(0,s) = \oint_C \frac{e^{-sz}}{\sinh^4 z}\,dz = \frac{i\pi}{3}(s+2)s(s-2),$$

where the contour C is including $\frac{\pi}{2}$ but excluding all zeros of $\cos z$. If $Y(0) = 0$, then $\lambda = -4$.

Proof. If $z = i\left(z - \frac{\pi}{2}\right)$, then

$$Y(0,s) = ie^{\frac{is\pi}{2}} \oint_C \frac{e^{-isz}}{\cos^4 z}\,dz = I_1 + I_2,$$

where

$$I_1 = ie^{\frac{is\pi}{2}} \oint_C \frac{\cos(sz)}{\cos^4 z}\,dz$$

$$= ie^{\frac{is\pi}{2}} f_4(s)$$

$$= ie^{\frac{is\pi}{2}} \oint_C \frac{\cos((s-1)z + z)}{\cos^4 z}\,dz$$

$$= ie^{\frac{is\pi}{2}} \oint_C \frac{\cos(s-1)z\cos z - \sin(s-1)z\sin z}{\cos^4 z}\,dz,$$

by partial integration,

$$= ie^{\frac{is\pi}{2}}\left(1 + \frac{s-1}{3}\right) f_3(s-1)$$

$$= ie^{\frac{is\pi}{2}} \frac{2+s}{3} f_3(s-1),$$

keep doing this argument we end up with

$$= ie^{\frac{is\pi}{2}} \frac{2+s}{3}\frac{s}{2}\frac{s-2}{1} f_1(s-3),$$

where

$$f_1(s-3) = \oint_C \frac{\cos(s-3)z}{\cos z}\,dz = 2\pi i\ \cos\frac{(s-3)\pi}{2}.$$

All in all,

$$I_1 = \frac{-2\pi e^{\frac{is\pi}{2}}}{3!}(2+s)s(s-2)\cos\frac{(s-3)\pi}{2}.$$

Similarly, since

$$I_2 = e^{\frac{is\pi}{2}} \oint_C \frac{\sin(sz)}{\cos^4 z} dz$$
$$= e^{\frac{is\pi}{2}} \widehat{f_4}(s)$$
$$= e^{\frac{is\pi}{2}} \oint_C \frac{\sin((s-1)z+z)}{\cos^4 z} dz$$
$$= e^{\frac{is\pi}{2}} \oint_C \frac{\sin(s-1)z\cos z + \cos(s-1)z \sin z}{\cos^4 z} dz,$$

partial integration implies that,

$$I_2 = e^{\frac{is\pi}{2}} \left(1 + \frac{s-1}{3}\right) \widehat{f_3}(s-1)$$
$$= e^{\frac{is\pi}{2}} \frac{2+s}{3} \widehat{f_3}(s-1),$$

keep doing this argument,

$$I_2 = e^{\frac{is\pi}{2}} \frac{2+s}{3} \frac{s}{2} \frac{s-2}{1} \widehat{f_1}(s-3),$$

where

$$\widehat{f_1}(s-3) = \oint_C \frac{\sin(s-3)z}{\cos z} dz = 2\pi i \sin \frac{(s-3)\pi}{2}.$$

So,

$$I_2 = \frac{2\pi i e^{\frac{is\pi}{2}}}{3!} (2+s)s(s-2) \sin \frac{(s-3)\pi}{2}.$$

Combining I_1 and I_2 , one obtains the expected result. If $Y(0) = 0$ implies that $s = \pm 2$ or $\lambda = -4$. That completes the proof.

Theorem 4

$$Y'(0,s) = 4 \oint_C \frac{e^{-sz}}{\sinh^5 z} dz = \frac{\pi}{3} (3+s)(1+s)(s-1)(s-3),$$

where the contour C is including $\frac{\pi}{2}$ but excluding all zeros of $\cos z$. If $Y'(0,s) = 0$, then $\lambda = -1, -9$.

Proof. Proof is the same as Theorem 3. Hence, it is excluded.

Theorem 5 $Y(x,s) \to 0$ as $x \to \infty$.

Proof. Set $z = iw$ and $x = iy$. So,

$$Y(y,\lambda) = i\cos^4 y \int \frac{e^{-isw}}{(\sin w - \sin y)^4} dw.$$
$$= i2^{-4}\cos^4 y \int \frac{e^{-isw}}{\left(\sin\frac{w-y}{2}\cos\frac{w+y}{2}\right)^4} dw$$
$$= i \int \frac{e^{-isw}}{(w-y)^4} dw.$$

By changing $w - y = u$ and $isu = v$ isu respectively, one obtains

$$Y(y,\lambda) = s^3 e^{-isy} \int e^{-v} v^{-4} \, dv$$
$$= s^3 e^{-sx} \int e^{-v} v^{-4} \, dv.$$

From here, it is trivial that this integral vanishes as x goes to ∞.

5. Eigenfunction Expansions

If $\theta(x,\lambda)$ and $\phi(x,\lambda)$ are the solutions of (2) and satisfying (3). Then one finds

$$\begin{cases}\phi(x,\lambda)=\frac{-\cos\alpha}{\lambda^2+10\lambda+9}y_1(x,\lambda)-\frac{\sin\alpha}{\sqrt{\lambda}\,(\lambda+4)}y_2(x,\lambda)\\ \theta(x,\lambda)=\frac{\sin\alpha}{\lambda^2+10\lambda+9}y_1(x,\lambda)-\frac{\cos\alpha}{\sqrt{\lambda}\,(\lambda+4)}y_2(x,\lambda).\end{cases} \tag{11}$$

Now, one needs to find $k(\lambda)$. To obtain the expected result, we need to get the asymptotics of solutions given in (11) as $x\to\infty$ such that $\Psi(x,\lambda)=\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)\in L^2(0,\infty)$. If $\mathrm{Im}\big(m(\lambda)\big)>0$, then the asymptotics are

$$\begin{aligned}\phi(x,\lambda)&\sim e^{-ix\sqrt{\lambda}}\left\{\frac{-\sqrt{\lambda}\,(\lambda+4)(6\lambda-6)\cos\alpha-\sqrt{\lambda}\,(11-\lambda)(\lambda^2+10\lambda+9)\sin\alpha}{2\sqrt{\lambda}\,(\lambda+4)(\lambda^2+10\lambda+9)}\right.\\ &\qquad\left.+i\frac{\lambda\,(\lambda+4)(11-\lambda)\cos\alpha-(6\lambda-6)(\lambda^2+10\lambda+9)\sin\alpha}{2\sqrt{\lambda}\,(\lambda+4)(\lambda^2+10\lambda+9)}\right\}\\ &=e^{-ix\sqrt{\lambda}}M_1(\lambda),\end{aligned} \tag{12}$$

and

$$\begin{aligned}\theta(x,\lambda)&\sim e^{-ix\sqrt{\lambda}}\left\{\frac{\sqrt{\lambda}\,(\lambda+4)(6\lambda-6)\sin\alpha-\sqrt{\lambda}\,(11-\lambda)(\lambda^2+10\lambda+9)\cos\alpha}{2\sqrt{\lambda}\,(\lambda+4)(\lambda^2+10\lambda+9)}\right.\\ &\qquad\left.+i\frac{-\lambda\,(\lambda+4)(11-\lambda)\sin\alpha-(6\lambda-6)(\lambda^2+10\lambda+9)\cos\alpha}{2\sqrt{\lambda}\,(\lambda+4)(\lambda^2+10\lambda+9)}\right\}\\ &=e^{-ix\sqrt{\lambda}}M(\lambda).\end{aligned} \tag{13}$$

After arranging the linear combination of (12) and (13), then the terms $e^{-ix\sqrt{\lambda}}$ cancel out. That is,

$$m(\lambda)=\begin{cases}-\dfrac{M(\lambda)}{M_1(\lambda)} & \text{if } \lambda>0\\ 0 & \text{if } \lambda<0,\end{cases}$$

where

$$\begin{aligned}M_1(\lambda)&=-\sqrt{\lambda}\,(\lambda+4)(6\lambda-6)\cos\alpha-\sqrt{\lambda}\,(11-\lambda)(\lambda^2+10\lambda+9)\sin\alpha\\ &\quad+i\{\lambda\,(\lambda+4)(11-\lambda)\cos\alpha-(6\lambda-6)(\lambda^2+10\lambda+9)\sin\alpha\}\end{aligned}$$

and

$$\begin{aligned}M(\lambda)&=\sqrt{\lambda}\,(\lambda+4)(6\lambda-6)\sin\alpha-\sqrt{\lambda}\,(11-\lambda)(\lambda^2+10\lambda+9)\cos\alpha\\ &\quad+i\{-\lambda\,(\lambda+4)(11-\lambda)\sin\alpha-(6\lambda-6)(\lambda^2+10\lambda+9)\cos\alpha\}.\end{aligned}$$

So that the continous spectrum is

$$k(\lambda)=-\mathrm{Im}(m(\lambda))=\begin{cases}\dfrac{\sqrt{\lambda}\,(\lambda+4)\,(\lambda^2+10\lambda+9)}{\lambda(\lambda+4)^2\cos^2\alpha+(\lambda^2+10\lambda+9)^2\sin^2\alpha} & \text{if } \lambda>0\\ 0 & \text{if } \lambda<0.\end{cases}$$

In particular, if $\alpha=0$ and $\lambda>0$, then one gets the continuous spectrum as

$$k(\lambda)=\begin{cases}\dfrac{(\lambda+1)(\lambda+9)}{\sqrt{\lambda}(\lambda+4)} & \text{if } \lambda>0\\ 0 & \text{if } \lambda<0.\end{cases}$$

Hence, the spectrum has only one eigenvalue occurring at $\lambda=-4$. From Lemma 1, one gets

$$f(x)=\frac{1}{\pi}\int_0^\infty \phi(x,\lambda)dk(\lambda)\int_0^\infty \phi(t,\lambda)f(t)\,dt+c_1\phi(x,-4),$$

where c_1 is constant,

$$\phi(x,\lambda)=\frac{\sqrt{\lambda}\sin\big(x\sqrt{\lambda}\big)\{-4+15\tanh^2 x-\lambda\}-\cos\big(x\sqrt{\lambda}\big)\{(9+6\lambda)\tanh x-15\tanh^3 x\}}{\lambda^2+10\lambda+9}$$

and

$$\phi(x,-4)=-\cosh(2x)\{\tanh x+\tanh^3 x\}+2\sinh(2x)\tanh^2 x.$$

If $\alpha=\frac{\pi}{2}$ and $\lambda>0$, then the spectrum is continuous

$$k(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}(\lambda+4)}{\lambda^2+10\lambda+9} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda < 0. \end{cases}$$

So that the spectrum has only two eigenvalues occurring at $\lambda = -1$ and $\lambda = -9$. From Lemma 1, one gets the following expansion

$$f(x) = \frac{1}{\pi}\int_0^\infty \phi(x,\lambda)dk(\lambda)\int_0^\infty \phi(t,\lambda)f(t)\,dt + c_1\phi(x,-1) + +c_1\phi(x,-9),$$

where c_1 and c_2 are constants. Here,

$$\phi(x,\lambda) = \frac{\sqrt{\lambda}\cos(x\sqrt{\lambda})\{4 - 15\tanh^2 x + \lambda\} - \sin(x\sqrt{\lambda})\{(9+6\lambda)\tanh x - 15\tanh^3 x\}}{\sqrt{\lambda}(\lambda+4)},$$

$$\phi(x,-1) = -4\operatorname{sech} x + 5\operatorname{sech}^3 x,$$

and

$$\phi(x,-9) = \cos\text{h}(3x)\{1 + 3\tanh^2 x\} - \sinh(3x)\{3\tanh x + \tanh^3 x\}.$$

Therefore, we have concluded the following theorem.

Theorem 6 If $\alpha = 0$, then there is just one spectral parameter (energetic eigenvalue) occurring at $\lambda = -4$ with energy function $\phi(x,-4)$. If $\alpha = \frac{\pi}{2}$, then there exist only two negative eigenvalues occurring at $\lambda = -1$ and $\lambda = -9$. So their associated spectral functions are given by $\phi(x,-1)$ and $\phi(x,-9)$.

We lastly consider the interval on $(-\infty,\infty)$ instead of positive interval. From (5) and (6) given in Lemma 2, since $q(x)$ is an even function of x, we obtain

$$\xi'(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}\,(\lambda+4)\,(\lambda^2+10\lambda+9)}{2\{\lambda(\lambda+4)^2\sin^2\alpha + (\lambda^2+10\lambda+9)^2\cos^2\alpha\}} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda < 0, \end{cases}$$

and

$$\zeta'(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}\,(\lambda+4)\,(\lambda^2+10\lambda+9)}{2\{\lambda(\lambda+4)^2\cos^2\alpha + (\lambda^2+10\lambda+9)^2\sin^2\alpha\}} & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda < 0. \end{cases}$$

Thus, from Lemma 2, one has the following expansion for square integrable function $f(x)$.

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\theta(x,\lambda)d\xi(\lambda)\int_{-\infty}^{\infty}\theta(t,\lambda)f(t)\,dt + \frac{1}{\pi}\int_{-\infty}^{\infty}\phi(x,\lambda)d\zeta(\lambda)\int_{-\infty}^{\infty}\phi(t,\lambda)f(t)\,dt$$

$$+c_1\phi(x,-4)+c_2\phi(x,-1) + c_3\phi(x,-9),$$

where c_1, c_2 and c_3 are constants. Here, $\phi(x,\lambda)$ and $\theta(x,\lambda)$ are given by (11) and, $\xi'(\lambda)$ and $\zeta'(\lambda)$ are given as above.

6. Conclusions

The explicit solutions of differential equation (2) and its eigenpairs (energey states and energy function) are obtained by calculating complex residues. Eigenfunction (energy function), expansions for arbitrary function $f(x)$ satisfying suitable conditions are also obtained. We concluded that the differential equation has oscillating solutions which are not reported in the literaure.

Acknowledgment

The author thanks to Guest Chief Editor Professor Mustafa BAYRAM, chairing ICAAMM, for his useful comments and encouragement related to publication of this paper.

Nomenclature

Greek Letters

$\boldsymbol{\theta}(\boldsymbol{x},\boldsymbol{\lambda})$*: Solution of Schrödinger equation or energy function*

$\boldsymbol{\lambda}$*: Energetic eigenvalues*

$\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{\lambda})$*: Solution of Schrödinger equation or energy function*

$\boldsymbol{\psi}$*: Square integrable solving Schrödinger equation*

References

[1] Titchmarsh, E. C., *Eigenfunction Expansions Associated with Second-Order Differential Equations Part I*, Oxford University Press, Oxford, UK, 1962

[2] Kamke, E., *Differentialgleichungen*, Akademische Verlagsgesellschaft, Leipzig, Germany, 1943

[3] Tanriverdi, T., *Boundary-Value Problems in ODE*, Ph.D. thesis, University of Pittsburgh, Pittsburgh, USA, 2001

[4] Tanriverdi, T., McLeod, J. B., Generalization of the eigenvalues by contour integrals, *Appl. Math. Comput.*, 189 (2007), 2, pp. 1765-1773

[5] Tanriverdi, T., McLeod, J. B., The Analysis of Contour Integrals, *Abstr. Appl. Anal.*, 2008 (2008), pp. 1-12

[6] Tanriverdi, T., Contour integrals associated differential equations, *Math. Comput. Modelling*, 49(2009), (3-4), pp. 453-462

[7] Tanriverdi, T., Differential equations with contour integrals, *Integral Transforms Spec. Funct.*, 20(2009), 2, pp. 119-125

[8] Everitt, W.N., *A catalogue of Sturm-Liouville differential equations*, Birkhauser Verlag, Basel, 2005, pp. 271-331

[9] Hahn, D. W., Ozisik, M. N., *Heat Conduction*, 3rd edition, Wiley, New York, USA, 2012

[10] Popović, M . E., There are two twin shadows, but Einstein is one, *Thermal Science*, (16) 2012, 1, pp. 1-6

[11] Wu, F., et al., Work output and efficiency of a reversible quantum otto cycle, *Thermal Science*, (14)2010, 4, pp. 879-886

[12] Krapez, J. C., Linear, trigonometric and hyperbolic profiles of thermal effusivity in the Liouville space and related quadrupoles: Simple analytical tools for modeling graded layers and multilayers, *International Journal of Thermal Sciences*, 136(2019), pp. 182-199