A NEW NOTION OF TRANSITIVE RELATIVE RETURN RATE AND ITS APPLICATIONS USING STOCHASTIC DIFFERENTIAL EQUATIONS

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We introduce a new notion of transitive relative return rate and present its applications based on the stochastic differential equations. First, we define the notion of a relative return rate (RRR) and show how to construct the transitive relative return rate (TRRR) on it. Then, we state some propositions and theorems about RRR and TRRR and prove them. Moreover, we exhibit the theoretical framework of the generalization of TRRR for $n \geq 3$ *cases and prove it, as well. Furthermore, we illustrate our approach with real data applications of daily relative return rates for Borsa Istanbul-30 (BIST-30) and Intel Corporation (INTC) indexes with respect to daily interest rate of Central Bank of the Republic of Turkey (CBRT) between 18.06.2003 and 17.06.2013. For this purpose, we perform simulations via Milstein method. We succeed to present usefulness of the relative return rate for the relevant real large data set using the numerical solution of the stochastic differential equations. The simulation results show that the proposed closely approximates the real data.*

Key words*: Stochastic differential equations, transitive relative return rate, Milstein method, numerical solution of stochastic differential equations*

1 Introduction

It is challenging to find a proxy between stochastic variables in the financial markets since it is hard to explain and control their relationship caused by the naturalness of them. Our aim in this paper is to present a proxy for return rates which helps to explain and understand their relative behavior with respect to the others. The relative return rate approximation can be useful for economists, investors or practitioners when they want to analyze and give an expectation for any security, whose return rate is not known at the time of its analysis, just by using the relative return rate instead of its return rate. For this purpose, we mathematically define the relative return rate and prove some propositions and theorems about this notion. Moreover, we extend this approximation to the transitive relative return rate which gives the approximate return rate value with respect to others. The most important aspect of this extension is that it may explain the relation between the three securities. After that, using the chain rule as a motivation we generalize transitive relative return rate for more than 3 securities so that the relative return rate of any security with respect to the security being considered can be obtained, indirectly.

Moreover, we believe that transitive relative return rate may have a wide application area in the literature, such as in physical and chemical process modelling and also in thermal sciences. For example, parabolic PDEs like Black-Scholes equation with a terminal condition are known to be related to the classical heat (diffusion) equation which models the evolution of the concentration of heat or chemical substances starting with an initial condition. In fact, using a sequence of transformations (change of variable), we can reduce the Black-Scholes PDE to the heat equation. Similarly, İzgi and Bakkaloğlu in [1] obtained transformations which reduce Black-Derman-Toy PDE to the third Lie canonical form using Lie symmetry analysis. Furthermore, such PDEs are also related to the random diffusion processes through the well-known Feynman-Kac formula. For more details, one can refer to Mao [2]. So, it might be interesting to explore the application of the relative return rates to the deterministic systems of diffusion equations with varying model parameters or initial conditions.

On the other hand, studies of the literature show that finding the relation between stochastic variables or deterministic time series has an important role in financial applications. Here, we present some of them among others, for example, in [3] Baker and Wurgler investigate the comovement and predictability of relationships between bonds and stocks returns and they also emphasize that it is difficult to understand the relationship between them which depends on the market's situation. In addition, Geweke [5] considers the measures of linear dependence and feedback for multiple time series in three directions. He advocates that these concepts can be useful for describing an estimated relationship and properties between two time series or econometric models. Duran and İzgi [4, 6] present a proxy for the time evolution of market impression, which displays and helps to understand the interrelation of stock price with stochastic volatility and interest rate, in the view of the impression matrix norms. In 2016, Chen et al. comprehensively revisit comovement behavior of indexes and stock splits by considering the two well-known paper's results in the literature [10]. In [11], Bunn et al. presented the effect of hedging and speculative activities onto the oil and gas prices using a large international dataset of real and nominal macro variables from emerging economies in 2017.

Furthermore, we present real data applications of relative return rates for Borsa Istanbul-30 (BIST-30) and Intel Corporation (INTC) indexes with respect to the Central Bank of the Republic of Turkey (CBRT). In these applications, we use daily relative return rates for BIST-30 and INTC with respect to daily interest rates of CBRT which we obtain by using daily return rates of BIST-30 and INTC indexes and daily interest rates of CBRT between $18.06.2003$ and $17.06.2013$. In addition, we perform simulations for these real data sets using relative return rates via a stochastic model based on the Black Scholes model [7] and show that our proxy, which is supported by the graphics we obtain from simulations by Milstein method [8] with the real data, is considerably suitable for financial markets. We believe that it may help investors or practitioners to catch behaviors or paths of the indexes with respect to time from the simulation results.

The remainder of the paper is organized as follows: In Section 2, we introduce the notion of a relative return rate and present the construction of transitive relative return rate on it. We also present some propositions and theorems about RRR and TRRR and prove them. In section 3, we prove the theorem which represents the generalization of TRRR for $n \geq 3$ and solve some examples. In Section 4, we examine daily relative return rate for BIST-100 and INTC indexes with respect to daily interest rate of CBRT between $18.06.2003$ and $17.06.2013$. We notice that this approximation produces an almost parallel pattern to the real data set. Section 5 concludes the paper.

2 Transitive Relative Return Rate (TRRR)

In this section, we first give the definition of a relative return rate and show how to construct the transitive relative return rate on it. Then, we present some propositions and theorems about TRRR and prove them. Moreover, we provide and solve some examples by using the theoretical results on transitive relative return rate which illustrate the consistency of the theoretical framework.

Definition 1. (Relative Return Rate (RRR)) The relative return rate (RRR) of a security X with respect to a security Y is:

$$RRR_{xy} = \frac{r_x - r_y}{r_y},\ r_y \neq 0$$

where r_x, r_y represent return rates of securities X and Y, respectively. On the other hand, RRR can be thought as a function which is defined from $\mathbb{R}^2$ except $y = 0$ line to the real numbers. (i.e. $RRR := \mathbb{R} \times \{\mathbb{R} \backslash \{0\}\} \to \mathbb{R}$).

Proposition 2. RRR is not a symmetric (i.e. $RRR_{xy} \neq RRR_{yx}$).
Proof. Since $RRR_{xy} = \frac{r_x - r_y}{r_y}$ and $RRR_{yx} = \frac{r_y - r_x}{r_x}$, we have

$$\begin{aligned}
&\frac{RRR_{xy}}{RRR_{yx}} = \frac{\frac{r_x - r_y}{r_y}}{\frac{r_y - r_x}{r_x}} = -\frac{r_x}{r_y} \\
&= 1 - 1 - \frac{r_x}{r_y} \\
&= -RRR_{xy} - 1 \\
&\Rightarrow RRR_{xy} = -\frac{RRR_{yx}}{1 + RRR_{yx}} \quad or \quad RRR_{xy} = (-1 - \frac{1}{RRR_{yx}})^{-1}
\end{aligned} \tag{1}$$

Corollary 3. Except the trivial case (i.e. $RRR = 0$ which implies $r_x = r_y$), RRR is symmetric ($RRR_{yx} = RRR_{xy}$) if and only if one of them is -2.

Definition 4 (Transitive Relative Return Rate (TRRR)). If we have 3 securities (X, Y, Z) and only know RRR_{XY} and RRR_{YZ} then relative return rate of X with respect to the Z, which can be computed with these values, is called transitive relative return rate and it is denoted by $TRRR_{XZ} (\equiv RRR_{XZ})$.

Theorem 5 (Transitivity of Relative Return Rate). Let RRR_{XY} be the relative return rate of a security X with respect to a security Y and RRR_{YZ} be the relative return rates of a security Y with respect to a security Z. Then, we can compute relative return rate of X with respect to the Z indirectly by,

$$TRRR_{XZ} = RRR_{XY} RRR_{YZ} + RRR_{XY} + RRR_{YZ} \tag{2}$$

Proof. Since RRR_{XY} and RRR_{YZ} are given then we can use their definitions directly such that $RRR_{XY} = \frac{r_X - r_Y}{r_Y}$ and $RRR_{YZ} = \frac{r_Y - r_Z}{r_Z}$. Then,

$$\begin{aligned}
RRR_{XY} RRR_{YZ} &= (\frac{r_X - r_Y}{r_Y})(\frac{r_Y - r_Z}{r_Z}) \\
&= \frac{r_X}{r_Z} - \frac{r_X}{r_Y} - \frac{r_Y}{r_Z} + 1,
\end{aligned}$$

$$= \underbrace{\frac{r_X}{r_Z} - 1}_{RRR_{XZ}} + \underbrace{1 - \frac{r_X}{r_Y}}_{-RRR_{XY}} + \underbrace{1 - \frac{r_Y}{r_Z}}_{-RRR_{YZ}}$$
$$\Rightarrow TRRR_{XZ} = RRR_{XY}RRR_{YZ} + RRR_{XY} + RRR_{YZ}$$

Example 6. Assume that we have 4 securities (X, Y, Z, T) and only know 3 $RRRs$ (i.e. $RRR_{XY}, RRR_{YZ}, RRR_{ZT}$), we can find relative return rate of X with respect to T by using Theorem 5, directly. If we apply formula (2) to the given RRRs then we get :

$$\begin{aligned}
&TRRR_{XT} = TRRR_{XZ}RRR_{ZT} + TRRR_{XZ} + RRR_{ZT} \\
&= (RRR_{XY}RRR_{YZ} + RRR_{XY} + RRR_{YZ})RRR_{ZT} + RRR_{XY}RRR_{YZ} \\
&+RRR_{XY} + RRR_{YZ} + RRR_{ZT} \\
&= RRR_{XY}RRR_{YZ}RRR_{ZT} + RRR_{XY}RRR_{YZ} + RRR_{XY}RRR_{ZT} \\
&+RRR_{YZ}RRR_{ZT} + RRR_{XY} + RRR_{YZ} + RRR_{ZT}
\end{aligned} \tag{3}$$

Proposition 7. If we have 3 securities (X, Y, Z) and only know RRR_{XY} and RRR_{ZY}, which are not in an order, then relative return rate of X with respect to the Z can be obtained by using the following formula:

$$TRRR_{XZ} = \frac{RRR_{XY} - RRR_{ZY}}{1 + RRR_{ZY}}.$$

Proof. The statement can be easily proved by using Proposition 2 and Theorem 5, directly.

Corollary 8. It is clear from Proposition 2 and Corollary 3 that $TRRR$ is not symmetric (i.e. $TRRR_{XZ} \neq TRRR_{ZX}$) in general except the following cases:

$$TRRR_{XZ} = \begin{cases} 0, & if\ RRR_{XY} = RRR_{ZY}; \\ -2, & if\ RRR_{XY} + RRR_{ZY} = -2. \end{cases}$$

3 Generalization of TRRR for $n \geq 3$ Cases

In this section, we give the generalization of the $TRRR$ for $n \geq 3$ securities whenever $n-1$ ordered $RRRs$ are known for $n \geq 3$ securities. The $n = 3$ case is identical to the classic $TRRR$ case which was considered explicitly in section 2. For the $n = 2$ and $n = 1$ cases the $TRRR$ converts to the RRR and return rate, respectively. On the other hand, we assume that none of the $TRRR_{ij}, 1 \leq i < j < n$ are known for the $n \geq 3$ case throughout this section. Otherwise, the new case(s) would be special case of the generalized one.

Definition 9. Let A be a finite non-empty subset of R. Then, $\overline{\sum}(X)$ is called an adding function of the element of A, and it is defined from A to $\mathbb{R}$ as follows:

$$\overline{\sum}(X): A \Rightarrow \mathbb{R}\ (i.e.\ \text{if}\ A = \{a, b, c\} \rightarrow \overline{\sum} A = \overline{\sum}\{a, b, c\} = a + b + c)$$

Definition 10. Let B be a finite non-empty subset of R. Then, $\overline{\prod}(X)$ is called a product function of the element of B, and it is defined from B to $\mathbb{R}$ as follows:

$$\overline{\prod}\,(X): B \Rightarrow \mathbb{R}\ (i.e.\ \text{if}\ B = \{a,b,c\} \to \overline{\prod}\,B = \overline{\prod}\,\{a,b,c\} = abc)$$

Theorem 11 (Generalization of TRRR). If we have $n \geq 3$ securities and know $n-1$ ordered $RRRs$ $(i.e.\ A = \{RRR_{12}, RRR_{23}, \ldots, RRR_{n-1n}\})$ then the transitive relative return rate of the first security with respect to the nth security can be obtained using the following formula:

$$TRRR_{1n} = \sum_k \big(\overline{\sum}\,\overline{\prod}^{(k)}\big),\ for\ \ k = 1, \ldots, n-1$$

where

$$\overline{\prod}^{(i)} = \overline{\prod}\,\{set\ of\ i-combination\ of\ A\}.$$

Proof. We will use mathematical induction on n to prove the statement:

1. $TRRR_{12} = \sum_{k=1}^{2-1} \big(\overline{\sum}\,\overline{\prod}^{(k)}\big) = RRR_{12}$ is obvious.
2. Assume that it holds for $n-1$ i.e. $TRRR_{1n-1} = \sum_{k=1}^{n-2} \big(\overline{\sum}\,\overline{\prod}^{(k)}\big)$. Let's show that it satisfies for n, too.
3. We want to show that $TRRR_{1n} = \sum_{k=1}^{n-1} \big(\overline{\sum}\,\overline{\prod}^{(k)}\big)$ holds. In this step, we use hat notation for $TRRR_{n-1n}$ to overcome the possible confusions, which may arise from representations, throughout operations.

$$TRRR_{1n} = TR\widehat{RR_{1n-1}}RRR_{n-1n} + TR\widehat{RR_{1n-1}} + RRR_{n-1n} \quad \text{(by Theorem 5)}$$

$$= \Big(\sum_{k=1}^{n-2} \big(\widehat{\overline{\sum}\,\overline{\prod}^{(k)}}\big)\Big)RRR_{n-1n} + \sum_{k=1}^{n-2} \big(\widehat{\overline{\sum}\,\overline{\prod}^{(k)}}\big) + RRR_{n-1n}$$

$$= \Big(\widehat{\overline{\sum}\,\overline{\prod}^{(1)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(2)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(3)}} + \ldots + \widehat{\overline{\sum}\,\overline{\prod}^{(n-2)}}\Big)RRR_{n-1n}$$

$$+ \sum_{k=1}^{n-2} \big(\widehat{\overline{\sum}\,\overline{\prod}^{(k)}}\big) + RRR_{n-1n}$$

$$= \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{2})} + \overline{\sum}\,\overline{\prod}^{(\mathbf{3})} + \overline{\sum}\,\overline{\prod}^{(\mathbf{4})} + \ldots + \overline{\sum}\,\overline{\prod}^{(\mathbf{n-1})}}_{\mathbf{A}}$$

$$+ \sum_{k=1}^{n-2} \big(\widehat{\overline{\sum}\,\overline{\prod}^{(k)}}\big) + RRR_{n-1n}$$

$$= \mathbf{A} + \widehat{\overline{\sum}\,\overline{\prod}^{(1)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(2)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(3)}} + \ldots + \widehat{\overline{\sum}\,\overline{\prod}^{(n-2)}} + RRR_{n-1n}$$

$$= \mathbf{A} + \overline{\sum}\,\overline{\prod}^{(\mathbf{1})} + \widehat{\overline{\sum}\,\overline{\prod}^{(2)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(3)}} + \ldots + \widehat{\overline{\sum}\,\overline{\prod}^{(n-2)}}$$

$$\text{(where } \widehat{\overline{\sum}\,\overline{\prod}^{(1)}} + RRR_{n-1n} = \overline{\sum}\,\overline{\prod}^{(\mathbf{1})}\text{)}$$

$$= \overline{\sum}\,\overline{\prod}^{(\mathbf{2})} + \overline{\sum}\,\overline{\prod}^{(\mathbf{3})} + \overline{\sum}\,\overline{\prod}^{(\mathbf{4})} + \ldots + \overline{\sum}\,\overline{\prod}^{(\mathbf{n-1})} + \overline{\sum}\,\overline{\prod}^{(\mathbf{1})}$$

$$+ \widehat{\overline{\sum}\,\overline{\prod}^{(2)}} + \widehat{\overline{\sum}\,\overline{\prod}^{(3)}} + \ldots + \widehat{\overline{\sum}\,\overline{\prod}^{(n-2)}}$$

$$= \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{1})}}_{\overline{\sum}\,\overline{\prod}^{(1)}} + \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{2})} + \widehat{\overline{\sum}\,\overline{\prod}^{(2)}}}_{\overline{\sum}\,\overline{\prod}^{(2)}} + \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{3})} + \widehat{\overline{\sum}\,\overline{\prod}^{(3)}}}_{\overline{\sum}\,\overline{\prod}^{(3)}} + \cdots$$

$$+ \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{n-2})} + \widehat{\overline{\sum}\,\overline{\prod}^{(n-2)}}}_{\overline{\sum}\,\overline{\prod}^{(n-2)}} + \underbrace{\overline{\sum}\,\overline{\prod}^{(\mathbf{n-1})}}_{\overline{\sum}\,\overline{\prod}^{(n-1)}}$$

$$= \overline{\sum}\, \overline{\prod}^{(1)} + \overline{\sum}\, \overline{\prod}^{(2)} + \overline{\sum}\, \overline{\prod}^{(3)} + \ldots + \overline{\sum}\, \overline{\prod}^{(n-2)} + \overline{\sum}\, \overline{\prod}^{(n-1)}$$

$$= \sum_k \left(\overline{\sum}\, \overline{\prod}^{(k)}\right), \quad \text{for } k = 1, 2, 3, \ldots, n-1.$$

Example 12. Solve Example 6 by using Theorem 11.

Solution: We define a set A such that $A = \{RRR_{XY}, RRR_{YZ}, RRR_{ZT}\}$ then $TRRR_{XT}$ can be determined with $TRRR_{XT} = \sum_{k=1}^{4-1} (\overline{\sum}\, \overline{\prod}^{(k)})$ by Theorem 11. In order to use this theorem directly first of all we need to obtain $\overline{\prod}^{(1)}, \overline{\prod}^{(2)}$ and $\overline{\prod}^{(3)}$. On the other hand, let $\{C\binom{n}{i}\}$ represents a set, which contains i- combination of a set for n elements. Here, the set is A and it has $n = 3$ elements, then

$$\{C\binom{3}{1}\} = \{\{RRR_{XY}\}, \{RRR_{YZ}\}, \{RRR_{ZT}\}\}$$

$$\{C\binom{3}{2}\} = \{\{RRR_{XY}, RRR_{YZ}\}, \{RRR_{XY}, RRR_{ZT}\}, \{RRR_{YZ}, RRR_{ZT}\}\}$$

$$\{C\binom{3}{3}\} = \{\{RRR_{XY}, RRR_{YZ}, RRR_{ZT}\}\}$$

Now, we are in the position to apply product function:

$$\overline{\prod}\,\{C\binom{3}{1}\} = \{RRR_{XY}, RRR_{YZ}, RRR_{ZT}\} = \overline{\prod}^{(1)}$$

$$\overline{\prod}\,\{C\binom{3}{2}\} = \{RRR_{XY}RRR_{YZ}, RRR_{XY}RRR_{ZT}, RRR_{YZ}RRR_{ZT}\} = \overline{\prod}^{(2)}$$

$$\overline{\prod}\,\{C\binom{3}{3}\} = \{RRR_{XY}RRR_{YZ}RRR_{ZT}\} = \overline{\prod}^{(3)}$$

Finally, we can apply adding function to the above sets and obtain results as follow:

$$TRRR_{XT} = \sum_{k=1}^{3} (\overline{\sum}\, \overline{\prod}^{(k)})$$

$$= \overline{\sum}\, \overline{\prod}^{(1)} + \overline{\sum}\, \overline{\prod}^{(2)} + \overline{\sum}\, \overline{\prod}^{(3)}$$

$$= RRR_{XY} + RRR_{YZ} + RRR_{ZT} + RRR_{XY}RRR_{YZ} + RRR_{XY}RRR_{ZT}$$

$$+ RRR_{YZ}RRR_{ZT} + RRR_{XY}RRR_{YZ}RRR_{ZT} \tag{4}$$

Note that, this result is consistent with the solution in equation (3) which is obtained for Example 6. In short, the results are equal in other words equations (3) and (4) are identical.

4 Real Data Applications Using RRR

Since it is hard to handle stochastic cases in the financial markets, real data application of stochastic differential equations have important role for risky players. For this purpose, we present real data applications of relative return rates for Borsa Istanbul-30 (BIST-30) and Intel Corporation (INTC) indexes with respect to Central Bank of the Republic of Turkey (CBRT). In particular, we use daily relative return rates for BIST-30 and INTC with respect to daily interest rates of CBRT which are obtained using daily return rates of BIST-30 and INTC indexes and daily interest rates of CBRT between 18.06.2003 and 17.06.2013. We show that the metric of relative return rate is useful for the relevant large data set, which includes 7827 observations in the data set, with the stochastic model based on Black-Scholes model [7] using the numerical solution of stochastic differential equations [9].

We consider the following stochastic differential equation for asset price process while we perform simulations via Milstein method [8, 9] for BIST-30, INTC and CBRT:

$$dS(t) = S(t)(\mu\nu + \nu)dt + \sigma\rho S(t)dW(t). \tag{5}$$

whose exact solution is

$$S(t) = S(0)exp((\mu\nu + \nu - \frac{1}{2}\sigma^2\rho^2)t + \sigma\rho W(t)).$$

In equation (5), the drift term $\mu\nu + \nu$ represents the expected return rate of $S(t)$ where μ and ν are constant. Moreover, W(t) is a standard one-dimensional Brownian motion, and the diffusion term σ represents volatility parameter of the asset price process.

Before we start applications of the relative return rate for real data, if we substitute $RRR_{yx}(t)$ and interest rate $r_x(t)$ for μ and ν, respectively, in the drift term of SDE in (5), we have;

$$dS(t) = S(t)(RRR_{yx}(t)r_x(t) + r_x(t))dt + \sigma\rho S(t)dW(t).$$

where ρ represents the correlation coefficient between the securities x and y.

For the first real data application, $RRR_{yx}(t)$ is the daily relative return rates of BIST-30 with respect to daily interest rate of CBRT, whereas for the second data application, $RRR_{yx}(t)$ represents the daily relative return rates of INTC with respect to daily interest rate of CBRT, and $r_x(t)$ is the daily interest rate of CBRT. For example, we choose the $S(0) = 13{,}740.86$ (BIST-30's value on 18.06.2003), $\rho = 0.033$ (correlation coefficient between daily return rates of BIST-30 and daily interest rates of CBRT) and the volatility parameter of BIST-30 as $\sigma = 0.1$ for the first application. Similarly, we choose the $S(0) = 21.75$ (INTC's value on 18.06.2003), $\rho = 0.02$ (correlation coefficient between daily return rates of INTC and daily interest rates of CBRT) and the volatility parameter of INTC as $\sigma = 0.1$ for the second application. We then perform 1000 simulations and obtain the graphs (see right panels of figure 1 and figure 2), which show the expected BIST-30 and INTC index values, using daily relative return rates between 2003 and 2013 for BIST-30 and INTC with respect to daily interest rates of CBRT.

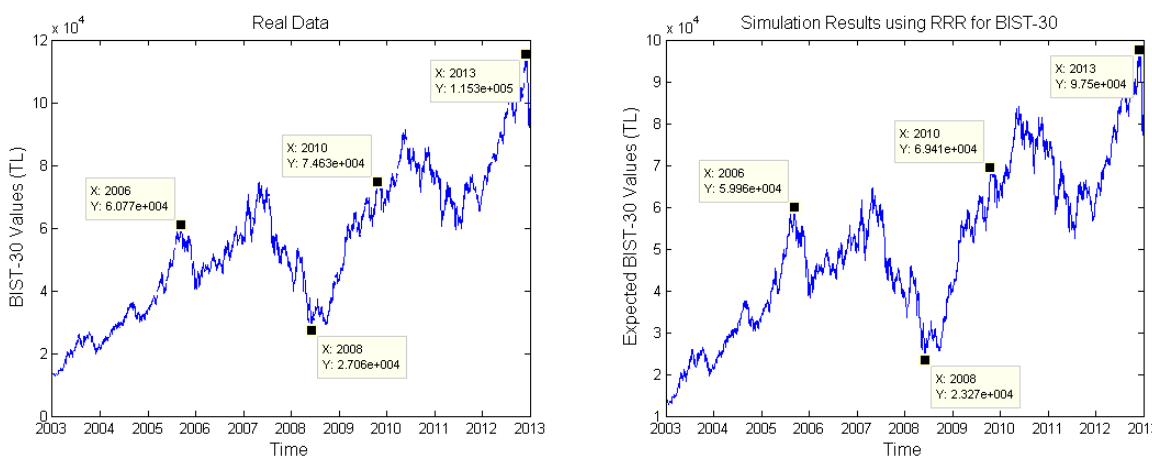

Figure 1: Real BIST-30 (left) and Expected BIST-30 (right) values between 2003 and 2013.

The right panel of the figure 1 and figure 2, which we obtain from simulations for BIST-30 and INTC indexes, look similar with the real data's graphs that are presented in the left panel of the figure 1 and figure 2, respectively. Although there are still approximation errors, these figures support that RRR approximation is useful to catch the pattern of the real data by performing simulations.

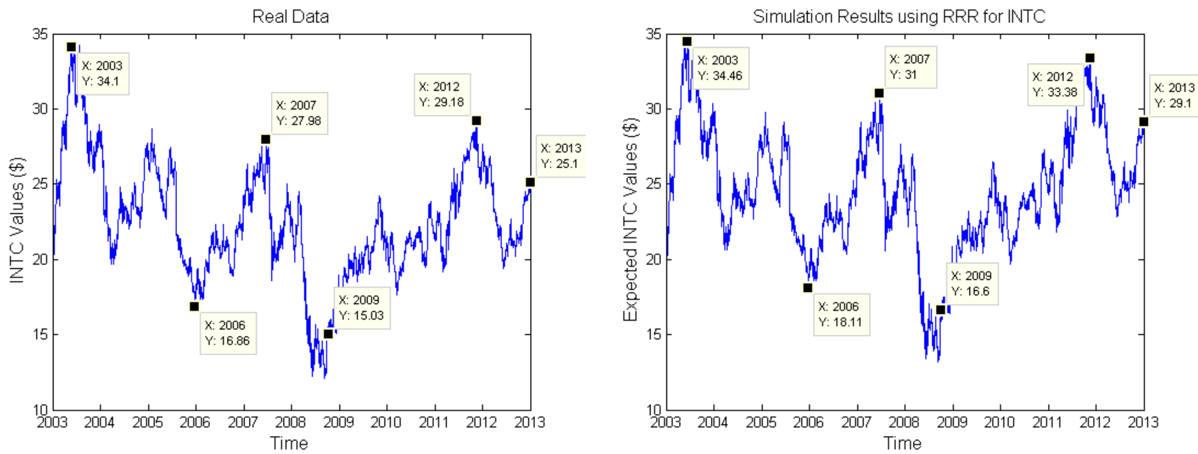

Figure 2: Real INTC (left) and Expected INTC (right) values between 2003 and 2013.

5 Conclusions

Recently, the approaches to the stochastic world via suitable models and methods have attracted academic attention in the literature. Although working with stochastic models is generally harder than deterministic ones, practitioners prefer to use stochastic models while they are modelling a problem since the stochastic models may reflect the real world behaviors better than deterministic models especially for the financial markets. We see that defining transitive and relative return rates is important and that the transitive and relative return rate approximation can be useful for economists, investors or practitioners when they want to analyze and give an expectation for any security.

Moreover, we present real data applications of daily relative return rates for BIST-30 and INTC indexes with respect to the daily interest rates of CBRT. We obtain important results and show that these approaches work by using the real data. While we compare the simulation results with the real data, we were successful in presenting very similar paths, which we obtained from simulation results for the first and second real data applications. These results show that relative return rate approximation is consistent with the real data and shows promise to be useful in different areas.

We believe that transitive relative return rate may attract academic attention in the literature since it may be used for defining or explaining the behavior of different securities with respect to the one being considered indirectly which is one of the most interesting and important uses of this proxy. Furthermore, we think that relative return rate approximation may also help to present comovement and polarization of securities.

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