MODIFIED LAGUERRE WAVELET BASED GALERKIN METHOD FOR FRACTIONAL AND FRACTIONAL-ORDER DELAY DIFFERENTIAL EQUATIONS

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The application of modified Laguerre wavelet with respect to the given conditions by Galerkin method to an approximate solution of fractional and fractional-order delay differential equations is studied in this paper. For the concept of fractional derivative is used Caputo sense by using Riemann-Liouville fractional integral operator.The presented method here is tested on several problems. The approximate solutions obtained by presented method are compared with the exact solutions and is shown to be a very efficient and powerful tool for obtaining approximate solutions of fractional and fractional-order delay differential equations . Some tables and figures are presented to reveal the performance of the presented method.

Keywords: *Galerkin method, Laguerre wavelet, Method of steps, Fractional differential equations, Fractional-order delay differential equations*

1. Introduction

Wavelets mean a family of piecewise functions which have many useful properties such as orthogonality, compact support, exact representation of polynomials to a certain degree and ability to represent fuctions at different levels of resolution [16]. Wavelets founded by Daubechies have been used to obtain approximate solution of different physical and mathematical problems related to various branches of engineering and applied sciences. Since the beginning of 1990s, wavelet based method have been used to solve different forms of differential equations. Generally, the wavelet coefficients have been found by using the Galerkin or collocation method. But Daubechies wavelet family has a drawback. Because these wavelet family have implicit expression, analytical differentiation or integration of Daubechies wavelets is not possible. And thus, more simple wavelets which are based on orthogonal polynomials such as Haar, Hermite, Legendre, Laguerre and Chebyshev are used in wavelet based numerical methods by many researchers [7-12].

Fractional-order delay differential equation is the generalization of delay differential equation. Fractional-order delay differantial equations and fractional differential equations have attracted increasing attention because fractional-order delay differantial equations and fractional differential equations have been applied in great various areas of physics, namely fluids mechanics [19], fluid-dynamic traffic [20], frequency dependent damping behavior of many viscoelastic materials, biosciences, signal processing [21] and control theory [22]. Moreover, fractional-order delay differantial equations and fractional differential equations are used in several areas of applied mathematics. These equations are also used in the study of epidemics, automation, traffic flow and in many engineering problems. So accurate and efficient numerical method for solution fractional and fractional-order delay differential equations is very substantial. To obtain numerical solutions of fractional and fractional-order delay differential equations , there are many numerical methods such as, Galerkin method, collocation method, operation matrix of integration method and operational matrix of differentiation etc, and different wavelets basis have been used. Modified Laguerre wavelets method are applied to delay differential equations of fractional- order [1]. The Legendre wavelet method is presented by Rehman and Khan [14]. Saeed and Rehman have solved by using Hermite wavelet method for delay differential equations of fractional- order [15]. Gegenbauer wavelets operational matrix method has been used to solve fractional differential equations in [17].

Laguerre wavelet based Galerkin method has been introduced for the numerical solution of elliptic problems. To basic aim of this paper is to develop Laguerre Wavelets Galerkin method and combine it with the method of steps to fractional and fractional-order delay differential equations.

2. Basic definitions of fractional calculus

Definition 1. The Riemann-Liouville fractional integral operator of order α is defined as [13]

$$I^{\alpha}u(t)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t \frac{u(s)}{(t-s)^{1-\alpha}}ds & ,\alpha>0\\ u(t) & ,\alpha=0\end{cases}$$

For the Riemann-Liouville fractional integral we have,

$$I^{\alpha}t^{\nu}=\frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\alpha)}t^{\nu+\alpha},\nu>-1.$$

The Riemann-Liouville fractional integral is a linear operation,

$$I^{\alpha}(\lambda u(t)+\mu v(t)=\lambda I^{\alpha}u(t)+\mu I^{\alpha}v(t),$$

where λ and μ are constants.

Definition 2. Caputo's fractional derivative of order α is defined as

$$D^{\alpha}u(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u^n(s)}{(t-s)^{\alpha+1-n}}ds, n-1<\alpha\le n, n\in \mathbf{N},$$

Where $\alpha>0$ is the order of the derivative and n is the smallest integer greater than α .

For Caputo's derivative we have the following four basic properties [14]

1. $D^{\alpha}I^{\alpha}u(t)=u(t)$
2. $I^{\alpha}D^{\alpha}u(t)=u(t)-\sum_{i=0}^{n-1}u^{(i)}(0)\dfrac{t^i}{i!}$
3. $D^{\alpha}t^{\nu}=\begin{cases}\dfrac{\Gamma(\nu+1)}{\nu+1-\alpha}t^{\nu-\alpha} & ,\alpha>0\\ 0 & ,\alpha\in 0,\nu<\alpha\end{cases}$
4. $D^{\alpha}\lambda=0$

where λ is constant.

3. Laguerre Wavelets

Wavelets form a family of functions which formulated from dilation and translation of a single function $\psi(t)$ called the mother wavelet. When the dilation parameter p and the translation parameter q vary continuously, we have the following family of continuous wavelets [1]:

$$\psi_{p,q}(t) = |p|^{-\frac{1}{2}} \psi\left(\frac{t-q}{p}\right), p, q \in R \text{ and } p \neq 0$$

If p and q parameters are restricted to discrete values as

$$p = p_0^{-k}, q = nq_0 p_0^{-k}, p_0 > 1, q_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |p|^{\frac{k}{2}} \psi(p_0^k t - nq_0), k, n \in Z$$

in which $\psi_{k,n}(t)$ form a wavelet basis for $L^2(R)$. If we take $p_0 = 2$ and $q_0 = 1$, then $\psi_{k,n}(t)$ forms an orthonormal basis.

The Laguerre wavelets $\psi_{n,m}(t) = \psi(k,n,m,t)$ comprise of four arguments, $n = 1,2,3,...,2^{k-1}$, m is the degree of Laguerre polynomials, and t is the normalized time. Laguerre wavelets are defined on the interval $[0,1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{L}_m(2^k t - 2n + 1) & , \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}} \\ 0 & , \text{otherwise} \end{cases} \quad (1)$$

where

$$\tilde{L}_m(t) = \frac{1}{m!} L_m(t) \quad (2)$$

$m = 0,1,2,..., M-1$ and M is any positive number integer. $L_m(t)$ are the Laguerre polynomials which are orthogonal with respect to the weight function $w(x) = 1$ on the interval $[0, \infty]$, and satisfy the following recursive formula

$$L_0(t) = 1, L_1(t) = 1 - t, L_{m+2}(t) = \frac{((2m+3-t)L_{m+1}(t) - (m+1)L_m(t)}{m+2}, m = 0,1,2,3,....$$

4. Modified Laguerre Wavelet Based Galerkin Method

In this paper, we take the following fractional-order delay differential equation of the form

$$u^{\alpha}(t) = g(t) + f(u(t), u'(t), u(qt-\tau), u'(qt-\tau)), a \leq t \leq b, 1 < a \leq 2, \quad (3)$$

$$u(t) = p(t), -b \leq t \leq a. \quad (4)$$

where $g(t)$ is a source function and f is a continuous linear or nonlinear function. q is constant, τ is delay, and $qt - \tau$ is defined delay argument. The delay $\tau(t, u(t))$ is defined constant delay, time dependent delay, and state dependent delay if the delay $\tau(t, u(t))$ is constant, function of time t, and function of time t and $u(t)$, respectively.

In this paper, presented method consists of two methods. One of two methods is method of steps, the other is Modified Laguerre Wavelet Based Galerkin method. Initially, we applied the method of steps to the fractional delay differential equation (3) and obtain the fractional nondelay differential equation by using initial function, $p(t)$. The obtained equation is solved by Laguerre wavelet Galerkin method.

4.1 Method of Steps

In the fractional-order delay differential equation the solution $u(t)$ is known on $[-b,a]$, be $p(t)$, and define this solution $u_0(t)$; that is $,u_0(qt-\tau)=p(qt-\tau)$, which is known. Now the fractional-order delay differential eqaution on $[a,b]$ is the following form

$$u^{\alpha}(t)=g(t)+f(u(t),u^{'}(t),u_0(qt-\tau),u_0^{'}(qt-\tau)),a\le t\le b,1<a\le 2. \quad (5)$$

with the initials conditions $u(a)=p(a),u^{'}(a)=p^{'}(a)$.

4.2 Modified Laguerre Wavelet Based Galerkin Method

The obtained fractional nondelay differential equation (5) on $[a,b]$ is solved by using the modified Laguerre wavelet based Galerkin method.

We can write the obtained fractional nondelay differential equation (5) as,

$$R(t)=u^{\alpha}(t)-g(t)-f(u(t),u^{'}(t),u_0(qt-\tau),u_0^{'}(qt-\tau)) \quad (6)$$

where $R(t)$ the residual of the obtained fractional nondelay differential equation (5). When $R(t)=0$ for the exact solution $u(t)$ which satisfy the given conditions. We consider the trail solution $u(t)$ for equation (5) can be expanded as a modified Laguerre wavelet [4] series with satisfying given conditions as follows:

$$u(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}a_{n,m}\psi_{n,m}(t) \quad (7)$$

where $a_{n,m}$'s are unknown coefficients to be determined. we differentitiate equation (7) order α with respect to t and substitute in the equation (6).To find the unknown coefficients $a_{n,m}$ in the equation (7), we choose weight functions as assumed basis elements and integrate on boundary values together with residual to zero (5). That is,

$$\int_0^1\psi_{n,m}(t)R(t)dt=0,n=1,..,2^{k-1},m=0,..,M-1. \quad (8)$$

Then we get system of linear equations. This system consists of $2^{k-1}M$ equations with unknown coefficients $a_{n,m}$. To determine the coefficients, this system must be solved. The determined coefficients $a_{n,m}$ are substituted into equation (7) and we can obtain the approximation solution.

5. Illustrative examples

Example 1. Consider the Fractional Delay Differential equation of the form

$$u^{\alpha}(t)+u(t)-u(t-\tau)=\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}-\frac{1}{\Gamma(2-\alpha)}t^{1-\alpha}+2\tau t-\tau^2-\tau,t>0,0<\alpha<1$$

$$u(t)=0,t\le 0$$

subject to the homogeneous boundary conditions $u(0)=u(1)=0$. This problem's exact solution [2] is $u(t)=t^2-t$ for $\alpha=1$. The modified Laguerre wavelets with respect to the given boundary conditions are considered as following;

$$\psi_{1,0}(t)=\sqrt{2}t\left(1-t\right)$$

$$\psi_{1,1}(t)=2\sqrt{2}t\left(1-t\right)$$

$$\psi_{1,2}(t)=1/4\sqrt{2}t(1-t)\left(4t^2-12t+7\right)$$

$$\psi_{1,3}(t) = 1/18\sqrt{2}t(1-t)\left(-4t^3 + 24t^2 - 39t + 17\right)$$

$$\psi_{1,4}(t) = 1/24\sqrt{2}t(1-t)\left(2/3t^4 - \frac{20}{3}t^3 + 21t^2 - \frac{73}{3}t + \frac{209}{24}\right)$$

$$\psi_{1,5}(t) = -\frac{1}{7200}\sqrt{2}t(1-t)\left(-773 + 2505t - 2720t^2 + 1240t^3 - 240t^4 + 16t^5\right)$$

$$\psi_{1,6}(t) = \frac{1}{518400}\sqrt{2}t(1-t)\left(62700t^2 - 48612t - 36640t^3 + 10320t^4 - 1344t^5 + 64t^6 + 13327\right)$$

$$\psi_{1,7}(t) = -\frac{1}{12700800}\sqrt{2}t(1-t)(-389592t^2 + 263431t + 274540t^3 - 100240t^4 + 19152t^5 - 1792t^6 + 64t^7 - 65461)$$

$$\psi_{1,8}(t) = \frac{1}{1625702400}\sqrt{2}t(1-t)(10448368t^2 - 6309648t - 8548288t^3 + 3800160t^4 - 947968t^5 + 130816t^6 - 9216t^7 + 256t^8 + 1441729)$$

$$\psi_{1,9}(t) = -\frac{1}{65840947200}\sqrt{2}t(1-t)(-75411072t^2 + 41368977t + 69812400t^3 - 36334368t^4 + 11106144t^5 - 2010624t^6 + 209664t^7 - 11520t^8 + 256t^9 - 8786057)$$

$$\psi_{1,10}(t) = \frac{1}{13168189440000}\sqrt{2}t(1-t)(2335600980t^2 - 1178821820t - 2401728960t^3 + 1424811360t^4 - 512797824t^5 + 114253440t^6 - 15682560t^7 + 1278720t^8 - 56320t^9 + 1024t^{10} + 234662231)$$

For $\alpha = 0.2, k = 1, M = 10$ and $\tau = 0.01e^{-t}$. The numerical results obtained by modified Laguerre wavelet based Galerkin method are shown in Table 1.

Table 1. Comparison of the solution by MLWGM at $M = 10, k = 1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	–0.09	–0.8999982968e–1	1.7032e–7
0.2	–0.16	–0.1600008206	8.206e–7
0.3	–0.21	–0.2099994312	5.688e–7
0.4	–0.24	–0.2399993404	6.596e–7
0.5	–0.25	–0.2500004260	4.260e–7
0.6	–0.24	–0.2400006480	6.480e–7
0.7	–0.21	–0.2099996862	3.138e–7
0.8	–0.16	–0.1599993387	6.613e–7
0.9	–0.09	–0.9000049039e–1	4.9039e–7
1.0	0	0	0

By taking $\alpha = 0.9$ and $\tau = 0.1$ in [18], the numerical results obtained by modified Laguerre wavelet based Galerkin method are shown in Table 2.

Table 2. Comparison of the solution by MLWGM at $M=10, k=1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	–0.09	$-0.9000013280e-1$	$1.3280e-7$
0.2	–0.16	–0.1600000775	$7.75e-8$
0.3	–0.21	–0.2100001793	$1.793e-7$
0.4	–0.24	–0.2400000961	$9.61e-8$
0.5	–0.25	–0.2500000053	$5.3e-9$
0.6	–0.24	–0.2400000626	$6.26e-8$
0.7	–0.21	–0.2100001190	$1.190e-7$
0.8	–0.16	–0.1600000346	$3.46e-8$
0.9	–0.09	$-0.9000000909e-1$	$9.09e-9$
1.0	0	0	0

Example 2. Consider the Fractional Delay Differential equation of the form

$$u^{\alpha}=\frac{3}{4}u(t)+u(\frac{t}{2})-t^{2}+2, 0\le t\le 1, 1<\alpha\le 2$$

with initial conditions $u(0)=0, u^{'}(0)=0$ .This problem's exact solution is $u(t)=t^{2}$. For $k=1, M=10, \alpha=1.99$, the comprasion of the absolute error between exact solution and the obtained solution by modified Laguerre wavelet based Galerkin method are shown in Table 3.

Table 3. Comparison of the solution by MLWGM at $M=10, k=1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	0.01	$0.9964132712e-2$	$3.5867288e-5$
0.2	0.04	$0.3994369179e-1$	$5.630821e-5$
0.3	0.09	$0.8995379096e-1$	$4.620904e-5$
0.4	0.16	0.1599737396	$2.62604-5$
0.5	0.25	0.2499960790	$3.9210e-6$
0.6	0.36	0.3600222114	$2.22114e-5$
0.7	0.49	0.4900502453	$5.02453e-5$
0.8	0.64	0.6400734438	$7.34438e-5$
0.9	0.81	0.8100879631	$8.79631e-5$
1.0	1.0	1.000095255	$9.5255e-5$

Example 3. Consider the Fractional Nonlinear Riccati Differential equation

$$u^{\alpha}(t)=-u^{2}(t)+1, 0<\alpha\le 1$$

subject to the initial condition $u(0)=0$.

The exact solution [3], when $\alpha = 1$, is $u(t) = \frac{e^{2t} - 1}{e^{2t} + 1}$. For $k = 1, M = 5$, the comprasion of the absolute error between exact solution and the obtained solution by modified Laguerre wavelet based Galerkin method are shown in Table 4.

Table 4. Comparison of the solution by MLWGM at $M = 5, k = 1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	$0.9966799456e-1$	$0.9953374757e-1$	$1.3424699e-4$
0.2	0.1973753203	0.1972857675	$8.95528e-5$
0.3	0.2913126124	0.2912490596	$6.35528e-5$
0.4	0.3799489622	0.3798718008	$7.71614e-5$
0.5	0.4621171572	0.4620241958	$9.29614e-5$
0.6	0.5370495670	0.5369653267	$8.42403e-5$
0.7	0.6043677771	0.6043100033	$5.77738e-5$
0.8	0.6640367702	0.6639956125	$4.11577e-5$
0.9	0.7162978702	0.7162489687	$4.89015e-5$
1.0	0.7615941560	0.7615531632	$4.09928e-5$

Example 4. Consider the following linear fractional differential equation of the form

$$u''(t) + t^2 u'(t) - u^{0.7}(t) + u(t) = f(t)$$

subject to the homogeneous boundary conditions $u(0) = u(1) = 0$. And $f(t) = 5t^6 - 3t^5 - t^4 + 20t^3 - 12t^2 - \frac{120}{\Gamma(5.3)} t^{4.3} + \frac{24}{\Gamma(4.3)} t^{3.3}$. This problem's exact solution [2] is $u(t) = t^4(t-1)$ for $\alpha = 1$. The numerical results obtained by modified Laguerre wavelet based Galerkin method are shown in Table 5.

Table 5. Comparison of the solution by MLWGM at $M = 20, k = 1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	–0.00009	$-0.8982575429e-4$	$1.7424571e-7$
0.2	–0.00128	$-0.1280218948e-2$	$2.18948e-7$
0.3	–0.00567	$-0.5670119995e-2$	$1.19995e-7$
0.4	–0.01536	$-0.1535940920e-1$	$5.9080e-7$
0.5	–0.03125	$-0.3125021352e-1$	$2.1352e-7$
0.6	–0.05184	$-0.5184078935e-1$	$7.8935e-7$
0.7	–0.07203	$-0.7202931982e-1$	$6.8018e-7$
0.8	–0.08192	$-0.8191947967e-1$	$5.2033e-7$
0.9	–0.06561	$-0.6561114710e-1$	$1.14710e-6$
1.0	0	0	0

Example 5. Consider the following linear fractional differential equation of the form

$$u''(t) + 0.5u^{0.3}(t) + u(t) = f(t)$$

subject to the homogeneous boundary conditions $u(0) = u(1) = 0$ [6]. And $f(t) = 4t^2(5t-3) + 0.5t^{3.7}\left(\frac{120}{\Gamma(5.7)}t - \frac{24}{\Gamma(4.7)}\right) + t^4(t-1)$. This problem's exact solution [2] is $u(t) = t^4(t-1)$ for $\alpha = 1$. The numerical results obtained by modified Laguerre wavelet based Galerkin method are shown in Table 6.

Table 6. Comparison of the solution by MLWGM at $M = 10, k = 1$ with exact solution

t	Exact solution	Solution by MLWGM	Error in MLWGM
0.1	–0.00009	$-0.9727441712e-4$	$7.27441712e-6$
0.2	–0.00128	$-0.1279192654e-2$	$8.07346e-7$
0.3	–0.00567	$-0.5660005116e-2$	$9.994884e-6$
0.4	–0.01536	$-0.1536281547e-1$	$2.81547e-6$
0.5	–0.03125	$-0.3126077218e-1$	$1.077218e-5$
0.6	–0.05184	$-0.5183870381e-1$	$1.29619e-6$
0.7	–0.07203	$-0.7201936870e-1$	$1.063130e-5$
0.8	–0.08192	$-0.8192218760e-1$	$2.18760e-6$
0.9	–0.06561	$-0.6561659713e-1$	$6.59713e-6$
1.0	0	0	0

6. Conclusion

In this study, we have been practically applied modified Laguerre wavelet based Galerkin method to solve fractional and fractional-order delay differential equations by using method of steps. The presented scheme is tested on some examples to see the accuracy and efficiency of presented method and the obtained numerical results are then compared with exact solutions. These comprasions reveal that the presented method is efficient and practically suited to find approximate solution the fractional and fractional-order delay differential equations. So the presented method is observed that can be an alternative way for the numerical solutions of the fractional and fractional-order delay differential equations. All of above numerical computations have been calculated using Maple software.

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