INTEGRAL BALANCE METHODS APPLIED TO NON-CLASSICAL STEFAN PROBLEMS

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We consider two different Stefan problems for a semi-infinite material for the non-classical heat equation with a source that depends on the heat flux at the fixed face. One of them, with constant temperature at the fixed face, was already studied in literature and the other, with a convective boundary condition at the fixed face, is presented in this work. Due to the complexity of the exact solution it is of interest to compare with approximate solutions obtained by applying heat balance integral methods, assuming a quadratic temperature profile in space. A dimensionless analysis is carried out by using the parameters: Stefan number and the generalized Biot number. In addition it is studied the case when Biot number goes to infinity, recovering the approximate solutions when a Dirichlet condition is imposed at the fixed face. Some numerical simulations are provided in order to verify the accuracy of the approximate methods.

Key words: Stefan problem, convective boundary condition, heat balance integral method, refined integral method, similarity solution

1 Introduction

Stefan problems model heat transfer processes that involve a change of phase. They constitute a broad field of study since they arise in a great number of mathematical and industrial significance problems [1]. A review on analytical solutions is given in [2]. In this paper, firstly, we consider a free boundary problem (P) with a non-classical heat equation for a semi-infinite material [3] defined by

$$\rho c\frac{\partial U}{\partial t} - k\frac{\partial^2 U}{\partial x^2} = -\gamma F\left(\frac{\partial U}{\partial x}(0,t),t\right) , \qquad 0 < x < S(t) , \qquad t > 0 , \tag{1}$$

$$U(0,t) = u_\infty > 0 , \qquad t > 0 , \tag{2}$$

$$U(S(t),t) = 0 , \qquad t > 0 , \tag{3}$$

$$k\frac{\partial U}{\partial x}(S(t),t) = -\rho l\dot{S}(t) , \qquad t > 0 , \tag{4}$$

$$S(0) = 0 . \tag{5}$$

where the thermal coefficients k, ρ, c, l and γ are positive constants and the control function F depend on the evolution of the heat flux at the boundary $x = 0$ as follows:

$$F\left(\frac{\partial U}{\partial x}(0,t),t\right) = \frac{\lambda_0}{t^{1/2}}\frac{\partial U}{\partial x}(0,t), \tag{6}$$

where $\lambda_0 > 0$ is a given constant. This problem was studied in [4].

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The phase-change problem is also considered with a convective condition [5] at the fixed face $x = 0$. It states that heat flux at the fixed face is proportional to the difference between the material temperature and the neighbourhood temperature, that is: $k\frac{\partial U}{\partial x}(0,t) = H(t)\left(U(0,t) - u_\infty\right)$, where $H(t)$ characterizes the heat transfer at the fixed face and $0 < U(0,t) < u_\infty$. We take a free boundary problem with a convective condition of the form $H(t) = \frac{h}{t^{1/2}}$ where $h > 0$ characterizes the heat transfer coefficients [6]. More precisely, we consider a free boundary problem (P_h) which is defined by eq. (1), conditions (3)-(5) of problem (P) and the condition

$$k\frac{\partial U}{\partial x}(0,t) = \frac{h}{t^{1/2}}(U(0,t) - u_\infty), \qquad t > 0\,, \tag{7}$$

instead condition (2) of problem (P).

Due to the non-linear nature of this type of problems exact solutions are limited to a few cases. Although it can be found exact solutions, it is usefull to solve them either numerically or approximately. Despite having the exact solution to the problem that we will study, it is very complicated to find the exact solution. The heat balance integral method introduced by Goodman in [7] is a well-known approximate mathematical technique for solving the location of the free front in heat-conduction problems involving a phase of change. This method consists in transforming the heat equation into an ordinary differential equation over time by assuming a quadratic temperature profile in space. In [8], [9]-[11] this method is applied using different accurate temperature profiles such as: exponential, potential, etc.

Recently, various papers has been published applying integral methods to a variety of thermal and free boundary problems, especially to non-linear heat conduction and fractional diffusion: [12]-[17].

In this paper, we obtain approximate solutions through heat balance integral methods and variants obtained thereof proposed in [18] for the problems (P) and (P_h). As one of the mechanisms for the heat conduction is the diffusion, the excitation at the fixed face $x = 0$ (for example, a temperature, a flux or a convective condition) does not spread instantaneously to the material $x > 0$. However, the effect of the fixed boundary condition can be perceived in a bounded interval $[0, \delta(t)]$ (for every time $t > 0$) outside of which the temperature remains equal to the initial temperature. The heat balance integral method presented in [7] established the existence of a function $\delta = \delta(t)$ that measures the depth of the thermal layer. In problems with a phase of change, this layer is assumed to be the free boundary, i.e $\delta(t) = s(t)$.

From condition (3), using eq. (1), we obtain the new condition:

$$\left(\frac{\partial U}{\partial x}\right)^2 (S(t),t) = -\frac{l}{kc}\left(k\frac{\partial^2 U}{\partial x^2}(S(t),t) - \gamma\frac{\lambda_0}{t^{1/2}}\frac{\partial U}{\partial x}(0,t)\right)\,. \tag{8}$$

From eq. (1) and conditions (3)-(4) we obtain the integral condition:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{S(t)} U(x,t)\,\mathrm{d}x \;=\; -\frac{\frac{\partial U}{\partial x}(0,t)}{\rho c}\left[\gamma\lambda_0\frac{S(t)}{t^{1/2}} + k\right] - \frac{l}{c}\dot{S}(t)\,. \tag{9}$$

The classical heat balance integral method introduced in [7] to solve problem (P) or (P_h) proposes the resolution of a problem that arises by replacing the eq. (1) by the condition (9), and the condition (4) by the condition (8), keeping all others conditions of the problem (P) or (P_h) equals.

In [18], a variant of the classical heat balance integral method was proposed by replacing only eq. (1) by condition (9), keeping all others conditions of the problem (P) or (P_h) equals.

From eq. (1) and conditions (2) and (3) we can also obtain:

$$\int_0^{S(t)}\int_0^x \frac{\partial U}{\partial t}(\xi,t)\mathrm{d}\xi\mathrm{d}x = \frac{1}{\rho c}\left[-\gamma\lambda_0\frac{S^2(t)}{2t^{1/2}}\frac{\partial U}{\partial x}(0,t)-ku_\infty-k\frac{\partial U}{\partial x}(0,t)S(t)\right]. \qquad (10)$$

The refined heat balance integral method introduced in [19] to solve the problem (P) proposes the resolution of the approximate problem that arises by replacing eq. (1) by condition (10), keeping all others conditions of the problem (P) or (P_h) equals.

For solving the approximate problems previously defined we propose a quadratic temperature profile in space as $U(x,t)=\tilde{A}u_\infty(1-\frac{x}{S(t)})+\tilde{B}u_\infty(1-\frac{x}{S(t)})^2$, $0<x<S(t)$, $t>0$, with $\tilde{A}$ and $\tilde{B}$ are unknown constants to be determined. Notice that U satisfies condition (3).

The goal of this paper is to study different approximations for one-dimensional one-phase Stefan problems with a source function that depends on the flux. It is considered two different problems, which differ from each other in the boundary condition imposed at the fixed face $x=0$: temperature (Dirichlet) condition or convective (Robin) condition. In Section 2 we present the exact solution of the problem (P) which was given in [4]. Taking advantage of the exact solution of (P), we obtain approximate solutions using the heat balance integral method, an alternative method of it and the refined integral method, comparing each approach with the exact one. A similar study is done in Section 3 for the problem with a convective condition at the fixed face, (P_h). In order to make this analysis, we obtain previously the exact solution of (P_h). We also study the limit cases of the obtained approximate solutions when $h\to\infty$, recovering the approximate solutions when a temperature condition at the fixed face is imposed.

2 Exact and approximate solutions to the one-phase Stefan problem for a non-classical heat equation with a source and a temperature condition at the fixed face

In this section we present the exact solution of the problem (P) and we obtain approximate solutions by using heat balance integral methods, comparing each approach with the exact one.

2.1 Exact solution of problem (P)

In [4], it has been proved that for each dimensionless parameter $\lambda=\frac{\gamma\lambda_0}{(k\rho c)^{1/2}}>0$, the free boundary problem (P), where F defined by (6), has a unique similarity solution of the type

$$u(x,t)=u_\infty\left(1-\frac{E(\eta,\lambda)}{E(\xi,\lambda)}\right) \quad , \quad 0<\eta=\frac{x}{2at^{1/2}}<\xi,$$
$$s(t)=2a\xi t^{1/2} \quad , \quad a^2=k/\rho c \ \text{(diffusion coefficient)},$$

where

$$E(x,\lambda)=\mathrm{erf}(x)+\frac{4\lambda}{\pi^{1/2}}\int_0^x f(r)\mathrm{d}r \quad , \quad f(x)=\exp(-x^2)\int_0^x\exp(r^2)\mathrm{d}r, \qquad (11)$$

and $\xi>0$ is the unique solution of

$$\mathrm{Ste}\ \exp(-x^2)-\pi^{1/2}\,x\,\mathrm{erf}(x)=2\lambda\left(2x\int_0^x f(r)\mathrm{d}r-\mathrm{Ste}\ f(x)\right) , \quad x>0, \qquad (12)$$

where $\mathrm{erf}(x) = \frac{2}{\pi^{1/2}} \int_0^x \exp(-r^2)\, \mathrm{d}r$, and the dimensionless parameter defined by $\mathrm{Ste} = \frac{c u_\infty}{l}$ represents the Stefan number. We remark that function f defined in (11), is called the Dawson's integral.

From now on, we will consider the case $\mathrm{Ste} \in (0,1)$, due to the fact that for most phase-change materials candidates over a realistic temperature, the Stefan number will not exceed one [20].

2.2 Approximate solution using the classical heat balance integral method

The classical heat balance integral method in order to solve the problem (P) proposes the resolution of the approximate problem (P_1) defined by (2), (3), (5), (8) and (9). Proposing the following quadratic temperature profile in space:

$$u_1(x,t) = A_1 u_\infty \left(1 - \frac{x}{s_1(t)}\right) + B_1 u_\infty \left(1 - \frac{x}{s_1(t)}\right)^2, \qquad 0 < x < s_1(t)\,, \quad t > 0\,,$$

the free boundary takes the form $s_1(t) = 2a\xi_1 t^{1/2}$, $t > 0$, where the constants A_1, B_1 and ξ_1 will be determined from the conditions (2), (8) and (9). We obtain:

$$A_1 = \frac{-2(3+\mathrm{Ste})\xi_1^2 + 12\lambda \mathrm{Ste}\, \xi_1 + 6\mathrm{Ste}}{\mathrm{Ste}\left(\xi_1^2 + 6\lambda\xi_1 + 3\right)}\,, \qquad B_1 = \frac{3(2+\mathrm{Ste})\xi_1^2 - 6\lambda\mathrm{Ste}\, \xi_1 - 3\mathrm{Ste}}{\mathrm{Ste}\left(\xi_1^2 + 6\lambda\xi_1 + 3\right)}\,,$$

and ξ_1 must be a positive solution of the polynomial equation:

$$-4\lambda\left(3 + 2\mathrm{Ste}\right) z^5 + 2\left(12 + 9\mathrm{Ste} + 2\mathrm{Ste}^2 - 12\lambda^2(3+2\mathrm{Ste})\right) z^4 - 12\lambda\left(-9 + 16\mathrm{Ste} + 4\mathrm{Ste}^4\right) z^3 + \\ + 12(1+2\mathrm{Ste})\left(-3 + (6\lambda^2 - 1)\mathrm{Ste}\right) z^2 + 72\lambda\mathrm{Ste}(1+2\mathrm{Ste})z + 18\mathrm{Ste} + 3\mathrm{Ste}^2 = 0\,, \quad z > 0\,. \tag{13}$$

It is easy to see that (13) has at least one solution. Descartes' rule of signs states that if the terms of a single-variable polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Therefore, in our case, to have a unique solution of (13) is enough to take λ such that $12 + 9\mathrm{Ste} + 2\mathrm{Ste}^2 - 12\lambda^2(3 + 2\mathrm{Ste}) < 0$, that is

$$\lambda > \left(\frac{2\mathrm{Ste}^2 + 9\mathrm{Ste} + 12}{36 + 24\mathrm{Ste}}\right)^{1/2} \equiv g(\mathrm{Ste}), \tag{14}$$

and as g is an increasing function then for $0 < \mathrm{Ste} < 1$ it is sufficient to take $\lambda > g(1) \cong 0.6191391874$.

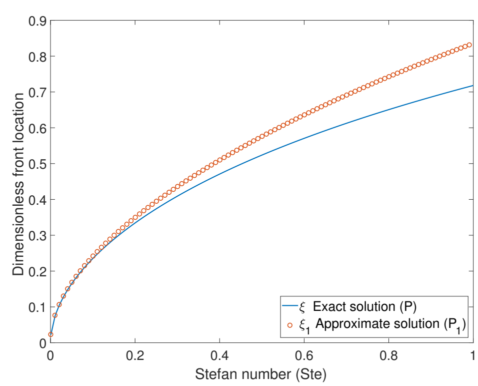

Figure 1 – Plot of the dimensionless coefficients ξ and ξ_1 against Ste number, for $\lambda = 0.7$.

As the approximate methods we are working with are designed as a technique for tracking the location of the free boundary, the comparisons between the approximate solutions and the exact one will be done on the free boundary thought the coefficients that characterizes them. That is to say, we will compare the known exact solution of the Stefan problem (P) and the approximate solution of the problem (P_1) by computing the coefficients ξ and ξ_1 that characterizes the free boundaries, which are obtained by solving (12) and (13), respectively. In fig. 1, we plot the dimensionless coefficients ξ and ξ_1 against Ste, fixing $\lambda = 0.7$.

2.3 Approximate solution using a modified method of the classical heat balance integral method

An alternative method of the classical heat balance integral method in order to solve the problem (P) proposes the resolution of the approximate problem (P_2) defined by (2)-(5) and (9).

Proposing the following quadratic temperature profile in space:

$$u_2(x,t) = A_2 u_\infty \left(1 - \frac{x}{s_2(t)}\right) + B_2 u_\infty \left(1 - \frac{x}{s_2(t)}\right)^2, \qquad 0 < x < s_2(t)\,, \quad t > 0\,,$$

the free boundary takes the form $s_2(t) = 2a\xi_2 t^{1/2}$, $t > 0$, where the constants A_2, B_2 and ξ_2 will be determined from the conditions (2), (4) and (9). We obtain:

$$A_2 = \frac{2}{\text{Ste}}\xi_2^2\,, \qquad B_2 = 1 - \frac{2}{\text{Ste}}\xi_2^2\,,$$

and ξ_2 must be a positive solution of the polynomial equation:

$$z^4 + 6\lambda z^3 + (6 + \text{Ste})\,z^2 - 6\lambda\text{Ste}\,z - 3\text{Ste} = 0\,, \qquad z > 0\,. \tag{15}$$

It is easy to see, using the Descartes' rule of signs, that (15) has a unique positive solution.

To compare the free boundaries obtained in problem (P) and the approximate problem (P_2), we compute the coefficient that characterizes the free boundaries. The exact value of ξ and the approach ξ_2 are the unique roots of equations (12) and (15), respectively.

In fig. 2 we show, for $0 < \text{Ste} < 1$, how the dimensionless coefficient ξ_2, which characterizes the location of the free boundary s_2, approaches the coefficient ξ, corresponding to the exact free boundary s, when the dimensionless parameter is $\lambda = 0.7$.

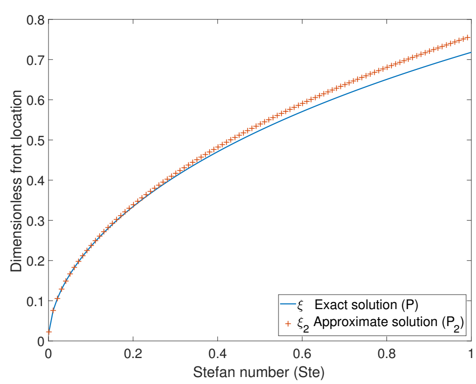

Figure 2 – **Plot of the dimensionless coefficients ξ and ξ_2 against Ste number, for $\lambda = 0.7$.**

2.4 Approximate solution using the refined integral method

The refined heat balance integral method in order to solve the problem (P), proposes the resolution of an approximate problem (P_3) formulated by conditions (2)-(5) and (10). Proposing the following quadratic temperature profile in space:

$$u_3(x,t) = A_3 u_\infty \left(1 - \frac{x}{s_3(t)}\right) + B_3 u_\infty \left(1 - \frac{x}{s_3(t)}\right)^2, \qquad 0 < x < s_3(t)\,, \quad t > 0\,,$$

the free boundary takes the form $s_3(t) = 2a\xi_3 t^{1/2}$, $t > 0$, where the constants A_3, B_3 and ξ_3 will be determined from the conditions (2), (4) and (10). We obtain:

$$A_3 = \frac{2}{\text{Ste}}\xi_3^2\,, \qquad B_3 = 1 - \frac{2}{\text{Ste}}\xi_3^2\,,$$

and ξ_3 must be a positive solution of the polynomial equation:

$$-6\lambda z^3 - (6 + \text{Ste})\, z^2 + 6\lambda \text{Ste}\, z + 3\text{Ste} = 0\,, \qquad z > 0\,. \tag{16}$$

It is easy to see, using the Descartes' rule of signs, that (16) has a unique positive solution.

To compare the free boundaries obtained in problem (P) and the approximate problem (P_3), we compute the coefficient that characterizes the free boundaries. The exact value of ξ and the approach ξ_3 is obtained by solving the equations obtained in (12) and (16), respectively.

For every $\text{Ste} < 1$, we plot the numerical value of the dimensionless coefficient ξ_3 obtained by applying the refined integral method, against the exact coefficient ξ (fig. 3).

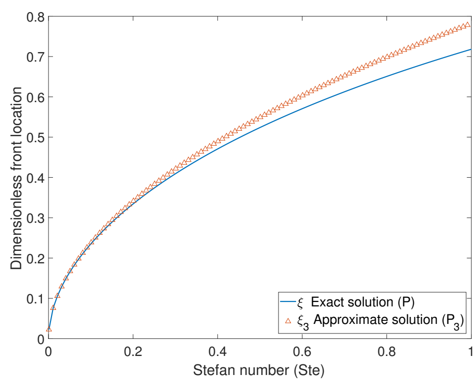

Figure 3 – Plot of the dimensionless coefficients ξ and ξ_3 against Ste number, for $\lambda = 0.7$.

2.5 Comparisons between solutions

In this subsection, for different Ste numbers, we make comparisons between the numerical value of the coefficient ξ given by (12) and the approximations ξ_1, ξ_2 and ξ_3 given by (13), (15), (16), respectively. In order to compare the approximate solution with the exact one, and to obtain which technique gives the best agreement, we display in tab. 1, the exact dimensionless free front, and its different approaches, showing also the percentage relative error committed in each case being $E_{\text{rel}}(\xi_i) = 100\frac{|\xi_i - \xi|}{|\xi|}$ $(i = 1, 2, 3)$.

It may be noticed that the relative error committed in each approximate technique increases when the Stefan number becomes greater, reaching the percentages 16.25% , 5.579% and 8.854% for the problems (P_1), (P_2) and (P_3) respectively.

Table 1 – Dimensionless free front coefficients and its relative errors for $\lambda = 0.7$.

Ste	ξ (P)	ξ_1 (P_1)	$E_{\rm rel}(\xi_1)$	ξ_2 (P_2)	$E_{\rm rel}(\xi_2)$	ξ_3 (P_3)	$E_{\rm rel}(\xi_3)$
0.1	0.2351	0.2401	2.139%	0.2363	0.545%	0.2373	0.963%
0.3	0.4091	0.4348	6.284%	0.4162	1.753%	0.4211	2.932%
0.5	0.5238	0.5750	9.776%	0.5392	2.934%	0.5489	4.788%
0.7	0.6125	0.6903	12.69%	0.6373	4.048%	0.6524	6.510%
0.9	0.6857	0.7897	15.16%	0.7206	5.087%	0.7413	8.102%

3 Exact and approximate solutions to the one-phase Stefan problem for a non-classical heat equation with a source and a convective condition at the fixed face

In this section we present the exact solution of the problem ($P_{\rm h}$) and we obtain different approaches by using heat balance integral methods, comparing them with the exact one.

3.1 Exact solution of problem $(P_{\rm h})$

In this subsection we will obtain the exact solution of the problem ($P_{\rm h}$) given by (1),(3)-(5) and (7) instead of condition (2) of problem (P). In similar way as [4], if we define the similarity variable $\eta = \frac{x}{2at^{1/2}}$ and $\Phi(\eta) = u_{\rm h}(x,t)$, then ($P_{\rm h}$) turns equivalent to the following ordinary differential problem:

$$\Phi''(\eta) + 2\eta\Phi'(\eta) = 2\lambda\Phi'(0), \qquad 0 < \eta < \xi_{\rm h} \tag{17}$$

$$\Phi'(0) = 2\text{Bi}\,(\Phi(0) - u_\infty) \tag{18}$$

$$\Phi(\xi_{\rm h}) = 0, \quad \Phi'(\xi_{\rm h}) = -2\tfrac{u_\infty}{\text{Ste}}\xi_{\rm h} \tag{19}$$

where the dimensionless parameter defined by $\text{Bi} = \frac{ha}{k}$ represent the generalized Biot number and $\xi_{\rm h}$ is the coefficient that characterizes the free boundary $s_{\rm h}$. It is a simple matter to find the solution to (17)-(19) and thus the solution to ($P_{\rm h}$) which is given by

$$u_{\rm h}(x,t) = \Phi(\eta) = \tfrac{\text{Bi}\ u_\infty \pi^{1/2}}{1+\text{Bi}\ \pi^{1/2} E(\xi_{\rm h},\lambda)}\left[E(\xi_{\rm h},\lambda) - E(\eta,\lambda)\right], \qquad 0 < \eta < \xi_{\rm h}$$

$$s_{\rm h}(t) = 2a\xi_{\rm h}t^{1/2}$$

where the function E is given by (11) and $\xi_{\rm h} > 0$ must be a solution of

$$\text{Ste}\ \exp(-x^2) - \pi^{1/2}\,x\,\text{erf}(x) - \frac{kx}{ha} = 2\lambda\left(2x\int_0^x f(r)\text{d}r - \text{Ste}\ f(x)\right), \quad x > 0. \tag{20}$$

We can apply similar results obtained in [4] to prove that there exists a unique solution $\xi_{\rm h}$ of (20).

Notice that in problem ($P_{\rm h}$) a convective boundary condition (7) characterized by the coefficient h at the fixed face $x = 0$ is imposed. This condition constitutes a generalization of the Dirichlet one in the sense that if we take de limit when $h \to \infty$ we must obtain $U(0,t) = u_\infty$. From definition of Bi, studying the limit behaviour of the solution to our problem ($P_{\rm h}$) when $h \to \infty$ is equivalent to study the case when Bi $\to \infty$.

If for every h, we define $\xi_{\rm h}$ as the unique solution to (20) then, it can be observed that $\{\xi_{\rm h}\}$ is increasing and bounded, and so convergent. In addition, it can be easily seen that $\xi_{\rm h} \to \xi$ where ξ is the unique solution to (12). Then, we can state that the solution to problem ($P_{\rm h}$) converges to the solution to problem (P) when Bi $\to \infty$ (i.e. $h \to \infty$), that is: $\lim_{h\to\infty} s_{\rm h}(t) = s(t)$ and $\lim_{h\to\infty} u_{\rm h}(x,t) = u(x,t)$, $0 < x < s(t),\ t > 0$.

3.2 Approximate solution using the classical heat balance integral method

The classical heat balance integral method in order to solve the problem $(P_{\rm h})$ proposes the resolution of the approximate problem $(P_{\rm h_1})$ defined by (3), (5), (7)-(9). For the quadratic temperature profile in space:

$$u_{\rm h_1}(x,t)=A_{\rm h_1}u_\infty\left(1-\frac{x}{s_{\rm h_1}(t)}\right)+B_{\rm h_1}u_\infty\left(1-\frac{x}{s_{\rm h_1}(t)}\right)^2,\qquad 0<x<s_{\rm h_1}(t)\,,\quad t>0\,,$$

the free boundary takes the form $s_{\rm h_1}(t)=2a\xi_{\rm h_1}t^{1/2},\ t>0$ where the constants $A_{\rm h_1}$, $B_{\rm h_1}$ and $\xi_{\rm h_1}$ will be determined from the conditions (7)-(9). We obtain:

$$A_{\rm h_1}=\frac{-2(3+{\rm Ste})\xi_{\rm h_1}^2+\left(12\lambda{\rm Ste}-\frac{6}{\rm Bi}\right)\xi_{\rm h_1}+6{\rm Ste}}{{\rm Ste}\left(\xi_{\rm h_1}^2+\left(6\lambda+\frac{2}{\rm Bi}\right)\xi_{\rm h_1}+3\right)}\,,\qquad B_{\rm h_1}=\frac{3(2+{\rm Ste})\xi_{\rm h_1}^2+\left(\frac{3}{\rm Bi}-6\lambda{\rm Ste}\right)\xi_{\rm h_1}-3{\rm Ste}}{{\rm Ste}\left(\xi_{\rm h_1}^2+\left(6\lambda+\frac{2}{\rm Bi}\right)\xi_{\rm h_1}+3\right)}\,,$$

and $\xi_{\rm h_1}>0$ must be a solution of the polynomial equation:

$$\begin{aligned}-4\lambda\,(3+2{\rm Ste})\,z^5+2\left(12+9{\rm Ste}+2{\rm Ste}^2-12\lambda^2(3+2{\rm Ste})-\tfrac{4\lambda}{\rm Bi}(3+2{\rm Ste})\right)z^4&+\\ +\left(-12\lambda(-9+16{\rm Ste}+4{\rm Ste}^4)+\tfrac{6}{\rm Bi}(7+2{\rm Ste})\right)z^3&+\\ +12\left((1+2{\rm Ste})(-3+(6\lambda^2-1){\rm Ste})+\tfrac{2}{{\rm Bi}^2}-\tfrac{3\lambda}{\rm Bi}(1+{\rm Ste})\right)z^2&+\\ +\left(72\lambda{\rm Ste}(1+2{\rm Ste})-\tfrac{6}{\rm Bi}(3+10{\rm Ste})\right)z+18{\rm Ste}+3{\rm Ste}^2=0\,,\qquad z>0\,.\end{aligned}\tag{21}$$

It is easy to see that eq. (21) has at least one solution. In order to prove uniqueness, we are going to use Descartes' rule of signs. Therefore, if we rewrite (21) as $\sum_{i=0}^5 a_iz^i=0$, we have to analyse the sign of each coefficient a_i. Clearly, $a_5<0$ and $a_0>0$. For $0<{\rm Ste}<1$ and $\lambda>0.62$, as in problem (P_1), $a_4<0$ for all Bi. Under these hypothesis: $a_3<0$ and $a_1>0$ if and only if ${\rm Bi}>\frac{3+10{\rm Ste}}{12\lambda{\rm Ste}(1+2{\rm Ste})}$.

Moreover the solution to problem $(P_{\rm h_1})$ converges to the solution to problema (P_1) when ${\rm Bi}\to\infty$.

To compare the solutions obtained in $(P_{\rm h})$ and $(P_{\rm h_1})$, we compute the coefficient that characterizes the free boundary in each problem. The exact value of ξ_h and the approach $\xi_{\rm h_1}$ are obtained by solving the equations obtained in (20) and (21), respectively. In fig. 4 we plot the coefficients $\xi_{\rm h}$ and $\xi_{\rm h_1}$ against Bi in order to visualize the behaviour of the approximate solution, fixing ${\rm Ste}=0.5$ and $\lambda=0.7$. In order that the convergence mentioned above of $\xi_{\rm h}\to\xi$ and $\xi_{\rm h_1}\to\xi_1$ when ${\rm Bi}\to\infty$, could be appreciated, we also plot ξ and ξ_1 given by the solution of (12) and (13), respectively.

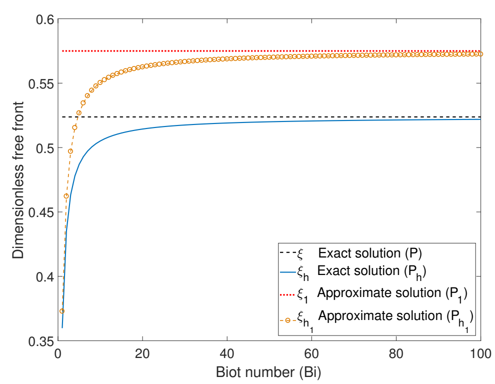

Figure 4 – Plot of the dimensionless coefficients $\xi_{\rm h}$ and $\xi_{\rm h_1}$ against Bi number, for Ste $=0.5$ and $\lambda=0.7$.

3.3 Approximate solution using a modified method of the classical heat balance method

An alternative method of the classical heat balance integral method in order to solve the problem $(P_{\rm h})$ proposes the resolution of the approximate problem $(P_{{\rm h}_2})$ defined by (3)-(5),(7) and (9). If

$$u_{{\rm h}_2}(x,t) = A_{{\rm h}_2} u_\infty \left(1 - \frac{x}{s_{{\rm h}_2}(t)}\right) + B_{{\rm h}_2} u_\infty \left(1 - \frac{x}{s_{{\rm h}_2}(t)}\right)^2, \qquad 0 < x < s_{{\rm h}_2}(t)\,, \quad t > 0\,,$$

then the free boundary takes the form $s_{{\rm h}_2}(t) = 2a\xi_{{\rm h}_2} t^{1/2}$, $t > 0$ where the constants $A_{{\rm h}_2}$, $B_{{\rm h}_2}$ and $\xi_{{\rm h}_2}$ will be determined from the conditions (4), (7) and (9). We obtain:

$$A_{{\rm h}_2} = \frac{2}{\rm Ste}\xi_{{\rm h}_2}^2\,, \qquad B_{{\rm h}_2} = \frac{-\frac{2}{\rm Ste}\xi_{{\rm h}_2}^3 - \frac{1}{\rm Ste\,Bi}\xi_{{\rm h}_2}^2 + \xi_{{\rm h}_2}}{\xi_{{\rm h}_2} + \frac{1}{\rm Bi}}\,,$$

and in this way, it turns out that $\xi_{{\rm h}_2} > 0$ must be a solution of the polynomial equation:

$$z^4 + \left(6\lambda + \tfrac{2}{\rm Bi}\right) z^3 + (6 + {\rm Ste})\, z^2 - \left(6\lambda{\rm Ste} + \tfrac{3}{\rm Bi}\right) z - 3{\rm Ste} = 0\,, \qquad z > 0\,, \tag{22}$$

where existence and uniqueness of solution for it can be easily seen by Descartes' rule of signs.

Moreover, the solution to problem $(P_{{\rm h}_2})$ converges to the solution to problem (P_2) when ${\rm Bi} \to \infty$.

Comparisons between the exact solution $\xi_{\rm h}$ with the approximate one $\xi_{{\rm h}_2}$ are shown in fig. 5. We plot them against Bi for ${\rm Ste} = 0.5$ and $\lambda = 0.7$. In order that the convergence of $\xi_{\rm h} \to \xi$ and $\xi_{{\rm h}_2} \to \xi_2$ when ${\rm Bi} \to \infty$, could be appreciated, we also plot ξ and ξ_2.

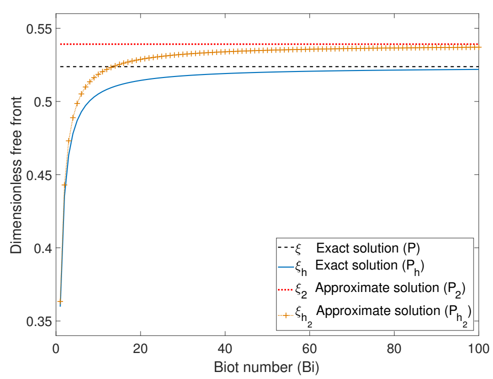

Figure 5 – Plot of the dimensionless coefficients $\xi_{\rm h}$ and $\xi_{{\rm h}_2}$ against Bi number, for Ste $= 0.5$ and $\lambda = 0.7$.

3.4 Approximate solution using the refined integral method

The refined heat balance integral method in order to solve the problem $(P_{\rm h})$, proposes de resolution of an approximate problem $(P_{{\rm h}_3})$ formulated by conditions: (3)-(5), (7) and (10). If we propose

$$u_{{\rm h}_3}(x,t) = A_{{\rm h}_3} u_\infty \left(1 - \frac{x}{s_{{\rm h}_3}(t)}\right) + B_{{\rm h}_3} u_\infty \left(1 - \frac{x}{s_{{\rm h}_3}(t)}\right)^2, \qquad 0 < x < s_{{\rm h}_3}(t)\,, \quad t > 0\,,$$

then the free boundary takes the form $s_{{\rm h}_3}(t) = 2a\xi_{{\rm h}_3} t^{1/2}$, $t > 0$, where the constants $A_{{\rm h}_3}$, $B_{{\rm h}_3}$ and $\xi_{{\rm h}_3}$ will be determined from the conditions (4), (7) and (10). We obtain:

$$A_{{\rm h}_3} = \frac{2}{\rm Ste}\xi_{{\rm h}_3}^2\,, \qquad B_{{\rm h}_3} = \frac{-\frac{2}{\rm Ste}\xi_{{\rm h}_3}^3 - \frac{1}{\rm Ste\,Bi}\xi_{{\rm h}_3}^2 + \xi_{{\rm h}_3}}{\xi_{{\rm h}_3} + \frac{1}{\rm Bi}}\,, \tag{23}$$

and $\xi_{\mathrm{h}_3} > 0$ must be a solution of the polynomial equation:

$$-\left(6\lambda + \frac{1}{\mathrm{Bi}}\right) z^3 - (6 + \mathrm{Ste})\, z^2 + \left(6\lambda \mathrm{Ste} - \frac{3}{\mathrm{Bi}}\right) z + 3\mathrm{Ste} = 0\,, \qquad z > 0\,. \tag{24}$$

Clearly, by Descartes' rule of signs, we can assure that (24) has a unique positive solution.

In addition, the solution to problem (P_{h_3}) converges to the solution to problem (P_3) when $\mathrm{Bi} \to \infty$.

In fig. 6, the coefficient that characterizes the free boundary of the exact solution ξ_{h} of problem (P_{h}) is compared with the coefficient ξ_{h_3} that characterizes the free boundary of the approximate problem (P_{h_3}), when we fix $\mathrm{Ste} = 0.5$ and $\lambda = 0.7$. We also show the value of ξ and ξ_3 in order to visualize the mentioned convergence when $\mathrm{Bi} \to \infty$.

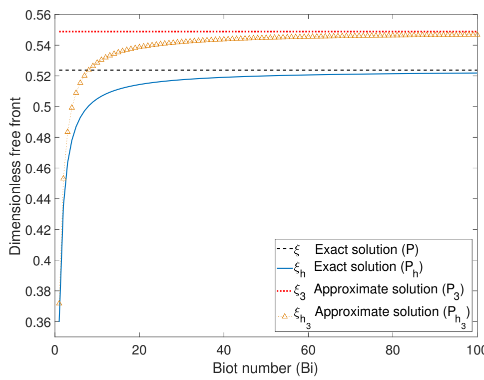

Figure 6 – Plot of the dimensionless coefficients ξ_{h} and ξ_{h_3} against Bi number, for Ste $= 0.5$ and $\lambda = 0.7$.

3.5 Comparisons between solutions

Let us compare, for different Bi numbers, the numerical value of the coefficient ξ_{h} given by (20) and the approximations ξ_{h_1}, ξ_{h_2} and ξ_{h_3} given by (21), (22), (24), respectively. In order to obtain which technique gives the best agreement, we display in tab. 2, varying Bi between 1 and 100, the exact dimensionless free front, and its different approaches, showing also the percentage relative error committed in each case $E_{\mathrm{rel}}(\xi_{\mathrm{h}_i}) = 100\frac{|\xi_{\mathrm{h}_i} - \xi_{\mathrm{h}}|}{|\xi_{\mathrm{h}}|}$.

Table 2 – Dimensionless free front coefficients and its relative errors.

Bi	ξ_{h} (P_{h})	ξ_{h_1} (P_{h_1})	$E_{\mathrm{rel}}(\xi_{\mathrm{h}_1})$	ξ_{h_2} (P_{h_2})	$E_{\mathrm{rel}}(\xi_{\mathrm{h}_2})$	ξ_{h_3} (P_{h_3})	$E_{\mathrm{rel}}(\xi_{\mathrm{h}_3})$
10	0.5051	0.5504	8.965%	0.5185	2.657%	0.5286	4.655%
30	0.5175	0.5667	9.501%	0.5322	2.840%	0.5421	4.745%
50	0.5200	0.5700	9.610%	0.5350	2.878%	0.5448	4.763%
70	0.5211	0.5714	9.657%	0.5362	2.894%	0.5459	4.770%
100	0.5219	0.5725	9.693%	0.5371	2.906%	0.5468	4.776%

From tab. 2, for the fixed values $\mathrm{Ste} = 0.5$ and $\lambda = 0.7$, we can appreciate that the error committed in each approximation increases when Bi becomes greater. We can notice that for the problems (P_{h_1}), (P_{h_2}) and (P_{h_3}) the percentage errors do not exceed 9.693%, 2.906% and 4.776% respectively.

4 Conclusion

In this paper we have considered two different Stefan problems for a semi-infinite material for the non-classical heat equation with a source which depends on the heat flux at the fixed face $x = 0$. The problem (P) with a prescribed constant temperature on $x = 0$ and the problem $(P_{\rm h})$ with a convective boundary condition at the fixed face which was studied in this article, proving existence and uniqueness of an exact solution. We have obtained, for $\lambda = 0.7$ that the best approximate solution to problem (P) was given by (P_2) obtaining a relative percentage error that does not exceed 5%. Furthermore the best approximation to problem $(P_{\rm h})$ was obtained by $(P_{{\rm h}_2})$ obtaining a relative error of 2.9%. Therefore it can be said that in general the optimal approximate technique for solving (P) and $(P_{\rm h})$ was given by the alternative form of the heat balance integral method, in which the Stefan condition is not removed and remains equal to the exact problem.

In addition it was studied the case when Bi goes to infinity in the solution to the exact problem $(P_{\rm h})$ an the approximate problems $(P_{{\rm h}_1})$, $(P_{{\rm h}_2})$, $(P_{{\rm h}_3})$, recovering the solutions to the exact problem (P) and the approximate problems (P_1), (P_2), (P_3). Some numerical simulations were also provided in order to visualize this asymptotic behaviour.

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Nomenclature

$a^2 = \frac{k}{\rho c}$	Thermal diffusivity, $[\rm m^2 s^{-1}]$	$\tilde{A}, \tilde{B}$	Coefficients in the prescribed temperature profile U, $[-]$
$A_{\rm i}, B_{\rm i}$	Coefficients in approximate temperature profiles $u_{\rm i}$, $(i = 1, 2, 3)$, $[-]$	$A_{{\rm h}_{\rm i}}, B_{{\rm h}_{\rm i}}$	Coefficients in approximate temperature profiles $u_{{\rm h}_{\rm i}}$, $(i = 1, 2, 3)$, $[-]$
$\text{Bi} = \frac{ha}{k}$	Biot number, $[-]$	c	Specific heat capacity, $[\rm J\ K^{-1} Kg^{-1}]$
E	Function defined by (11), $[-]$	F	Control function defined by (6), $[-]$
f	Function defined by (11), $[-]$	h	Heat transfer coefficient at the fixed face in, condition (7), $[\rm Ws^{1/2} m^{-2} K]$
g	Function defined by (14), $[-]$		
k	Thermal conductivity, $[\rm WK^{-1} m^{-1}]$	l	Latent heat per unit mass, $[\rm J\ Kg^{-1}]$
S	Free boundary, $[m]$	$s, s_{\rm h}$,	Free boundaries, solutions to problems (P) and $(P_{\rm h})$, respectively, [m]
$s_{\rm i}, s_{{\rm h}_{\rm i}}$,	Approximate free boundaries to problems (P) and $(P_{\rm h})$, respectively, $(i = 1, 2, 3)$, [m]	$\text{Ste} = \frac{cu_\infty}{l}$	Stefan number, $[-]$
t	Time, $[s]$	U	Temperature, [K]
u_∞	Bulk temperature at the fixed face condition 7, [K]	$u, u_{\rm h}$	Exact temperature solutions to problems (P) and $(P_{\rm h})$, respectively, [K]
$u_{\rm i}, u_{{\rm h}_{\rm i}}$	Approximate temperature solutions to problems (P) and $(P_{\rm h})$, respectively, [K]	x	Space coordinate, [m]
Greek symbols			
γ	Coefficient in eq. (1), $[\rm Wm^{-3}]$	$\eta = \frac{x}{2at^{1/2}}$	Similarity variable, $[-]$
λ_0	Coefficient that characterizes the control function F, $[\rm m\ s^{1/2} K^{-1}]$	$\lambda = \frac{\gamma \lambda_0}{(k\rho c)^{1/2}}$	Dimensionless coefficient, $[-]$
$\xi, \xi_{\rm h}$	Coefficients that characterizes the free boundaries s and $s_{\rm h}$, respectively, $[-]$	$\xi_{\rm i}, \xi_{{\rm h}_{\rm i}}$	Coefficients that characterizes the free boundaries $s_{\rm i}$ and$s_{{\rm h}_{\rm i}}$, $(i = 1, 2, 3)$, $[-]$
ρ	Mass density, $[\rm Kg\ m^{-3}]$	Φ	Function, solution of the ordinary differential problem (17-19), $[-]$

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