HEAT TRANSFER ANALYSIS IN MAGNETOHYDRODYNAMIC THERMAL NANOFLUID USING KELLER-BOX METHOD

by

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Thermal radiation analysis in MHD Casson nanofluid-flow over an exponentially stretching sheet is investigated. A chemical reaction is also considered. A non-uniform magnetic field of strength is imposed in a transverse direction. The governing boundary-layer equations are reduced into ODE by using suitable similarity transformations. The coupled non-linear equations are solved numerically using an implicit finite difference scheme by means of the Keller-box method. A comparison of the obtained results is performed with the published results. It is found that velocity profiles are suppressed with the increasing values of Hartmann number and Casson fluid parameter.

Key words: thermal nanofluid, radiation, MHD, chemical reaction, stretching sheet, Keller-box method

Introduction

In nanofluid studies the idea of Buongiorno [1] has received special attention specially from theoretical researchers. He studied nanofluid to increase its thermal conductivity in comparison with the base fluid and found that the Brownian motion and thermophoresis effects in the base fluid enhances the thermal conductivity of the liquid. Khan and Pop [2] firstly investigated the flow of the nanofluid together with Brownian and thermophoresis motion on the stretching surface. Yu et al. [3] summarized and lightened progress on the study of nanofluids and opportunities for future research such as the preparation methods, the evaluation methods for the stability of nanofluids and the ways to enhance the stability for nanofluids, the stability mechanisms of nanofluids, etc. Matsumi and Makinde et al. [4] studied numerically the effect of suction, viscous dissipation, thermal radiation and thermal diffusion on the boundary-layer flow of the nanofluid over a moving plate. Ravi et al. [5] discussed the various effects of parameters like particle size, volume fraction, material, etc. and different mechanisms to enhance the heat transfer qualities for Brownian motion, thermophoresis and clustering of nanoparticles.

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The Keller Box scheme for the solution of parabolic boundary-layer equations is both accurate and robust. It has been extensively used in solving broad class of problems including convection flows, jet flows, turbulent boundary-layer as well as separating flows. Keller box scheme has advantages in mathematics and physics:

- This scheme is implicit with 2nd order accuracy.
- It is efficient for the parabolic PDE.
- The accuracy of this method has been studied for incompressible and compressible, laminar and turbulent boundary-layer past 2-D and axisymmetric bodies.
- Keller box can be advantageous for security issues in networking.

Motivated by the previous investigations, the present work concerned with the chemical reaction and radiation effects on MHD Casson nanoluid-flow over an exponentially stretching sheet. By applying the suitable similarity transformations, the system of non-linear PDE are reduced into the system of non-linear ODE. Non-dimensional physical parameters namely Casson fluid parameter, Hartmann number, radiation parameter, chemical reaction parameter, Prandtl number, thermophoresis parameter, Brownian motion parameter, and Lewis number appear after reduction along with the system of coupled non-linear ODE. Coupled equations are then solved numerically by using Keller box method. A comparison of the present work with the previously published result and the behavior of each physical parameter are shown through tables and graphs.

**Problem formulation**

Consider a steady, viscous, incompressible, 2-D boundary-layer flow of Casson nanoluid over an exponentially stretching sheet. The stretching and free stream velocities are assumed to be of the forms \( u_\infty(x) = a \exp(\frac{x}{l}) \) and \( u_\infty(x) = 0 \), respectively, where \( a \) is constant, \( x \) – the co-ordinate measured along the stretching surface, and \( l \) – the length of the sheet. The temperature, \( T \), and the nanoparticles fraction, \( C \), take constant values \( T_w \) and \( C_w \), respectively, at the wall, whereas the ambient values of temperature, \( T_\infty \), and the
nanoparticles fraction $C_n$ are attained as $y$ tends to infinity. A non-uniform magnetic field of strength $B(x) = B_0 \exp(x/2l)$ is imposed in transverse direction (normal to the flow direction), where $B_0$ is the uniform magnetic field strength. It is assumed here that the induced magnetic field due to the motion of an electrically conducting fluid is negligible. Further, external electrical field is zero and the electric field due to the polarization of charges is negligible.

The governing boundary-layer equations:

$$\frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = 0 \quad (1)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \frac{1}{\rho_f} \frac{\partial q_r}{\partial y} + \left[ D_b \frac{\partial C}{\partial y} + D_T \frac{\partial T}{\partial y} \right] + \frac{\mu}{\rho_f} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u}{\partial y} \right)^2 \quad (2)$$

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} = D_b \frac{\partial^2 C}{\partial y^2} + D_T \frac{\partial^2 T}{\partial y^2} - K_1 (C - C_n) \quad (3)$$

where $u$ and $v$ are the velocity components along the $x$- and $y$-directions, respectively, $v$ is the kinematic viscosity, $\sigma$ – the electrical conductivity, $B_0$ – the strength of the magnetic field, $C$ – the species concentration in the base fluid, $\rho_f$ – the fluid density of the base fluid, $\alpha = k(\rho c_p)$ – the thermal diffusivity, $D_b$ – the Brownian diffusion coefficient, $D_T$ – the thermophoretic diffusion coefficient, $K_1$ – the chemical reaction parameter, and $\tau = (\rho c_p)/(\rho c_f)$, is the ratio of effective heat capacity of the nanoparticle material to the effective heat capacity of the base fluid:

$$\beta = \frac{\mu p^2 \Gamma c}{p_r}$$

is Casson fluid parameter (Non-Newtonian parameter). The $\Pi$, is the critical value of the product of the strain tensor with itself. In case of Casson fluid-flow $\Pi > \Pi$, [14, 15] as the particular amount of stress is to be applied to move Casson fluid such as tooth paste, honey, jelly, etc. Where $p_r$ is known as yield stress of the fluid and mathematically can be expressed:

$$p_r = \frac{\mu_b \sqrt{2\pi c}}{\beta}$$

where $\mu_b$ is plastic dynamic viscosity of the fluid. Casson fluid exhibit the yield stress as it is the property of elastic fluids. If the shear stress is less than the yield stress applied to it then Casson fluid behaves like a solid whereas if the shear stress is greater than the yield stress then it starts to move.

Here $\phi_r$ is the viscous dissipation function defined:

$$\phi_r = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} + \frac{\partial w}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)^2 - \frac{2}{3} \left( \nabla u \right)^2$$

where the Rosseland approximation (radiation flux) is defined:

$$q_r = -\frac{4\sigma^2}{3k} \frac{\partial T^4}{\partial y} \quad (5)$$
where $\sigma^*$ is the Stefan-Boltzmann constant and $k^*$ – the mean absorption coefficient. It is assumed that temperature difference between the free steam $T_\infty$ and local temperature $T$ is small enough, expanding $T^4$ in Taylor series about $T_\infty$, and neglecting higher order terms results:

$$T^4 \cong 4T_\infty^4 + T - 3T_\infty^4$$  \hspace{1cm} (6)

After substituting eqs. (9) and (10), eq. (7) reduces:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left( \alpha + \frac{16\sigma^*}{3k^*(\rho c)_f} \right) \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho c_p} \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u}{\partial y} \right)^2$$ \hspace{1cm} (7)

The subjected boundary conditions to this problem:

$$u = u_\infty(x) = ae^{x/l}, \quad v = 0, \quad T = T_\infty(x), \quad C = C_\infty(x) \quad \text{at} \quad y = 0$$
$$u \to 0, \quad v \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty$$ \hspace{1cm} (8)

The prescribed temperature and concentration on the surface of stretching sheet are assumed to be of the form $T_\infty(x) = T_\infty + T_0 \exp(x/2l)$ and $C_\infty(x) = C_\infty + C_0 \exp(x/2l)$ where $T_\infty$ and $C_\infty$ are the reference temperature and concentration, respectively.

The non-linear PDE are reduced into non-linear ODE. For the sake of this purpose the stream function $\psi = \psi(x, y)$ is defined:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$ \hspace{1cm} (9)

where the continuity eq. (1) is satisfied identically. Using the similarity transformations:

$$\psi = \sqrt{2\nu a} e^{x/2l} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty - T_\infty}, \quad \phi(\eta) = \frac{C - C_\infty}{C_\infty - C_\infty}$$
$$\eta = \sqrt{\frac{a}{2\nu \ell}} e^{x/2l}$$ \hspace{1cm} (10)

On substituting eqs. (9) and (10) in to eqs. (2), (4), and (7) reduces to the non-linear ODE:

$$\left( 1 + \frac{1}{\beta} \right) f''' + ff'' - 2f'^2 - Mf'' = 0$$ \hspace{1cm} (11)

$$Pr_\nu \theta'' + f \theta' - f' \theta + Ec \left( 1 + \frac{1}{\beta} \right) f'' + Nb \theta' \phi' + Nt \theta'' = 0$$ \hspace{1cm} (12)

$$\phi'' + Le f \phi' - Le \phi' + Nb \phi'' + Le \theta'' = 0$$ \hspace{1cm} (13)

where

$$\nu = \frac{\mu}{\rho_f}, \quad Pr = \frac{\nu}{\alpha}, \quad Le = \frac{\nu}{D_b}, \quad Ec = \frac{\nu^2}{(\rho c)_f (T_\infty - T_\infty)}, \quad Nt = \frac{Nt}{Nb}$$
$$Nb = \frac{\tau D_b (C_\infty - C_\infty)}{\nu}, \quad Nt = \frac{\tau D_b (T_\infty - T_\infty)}{\nu T_\infty}, \quad R = \frac{2lK_e}{u}$$
$$M = \frac{2l \sigma \beta^2}{a \rho_f}, \quad Pr_\nu = \frac{1}{Pr} \left( 1 + \frac{4}{3} N \right), \quad N = \frac{4 \sigma^* T_\infty^4}{kk}$$ \hspace{1cm} (14)
where prime denote the differentiation with respect to $\eta$, $v$ is the kinematic viscosity of the fluid, $Pr$ – the Prandtl number, $Le$ – the Lewis number, $Nb$ – the Brownian motion parameter, $Nt$ – the thermophoresis parameter, $M$ – Hartmann number, $N$ – the radiation parameter, $R$ – chemical reaction parameter and the corresponding boundary conditions (8) are transformed:

$$f(\eta) = 0, \ f'(\eta) = 1, \ \theta(\eta) = 1, \ \phi(\eta) = 1 \ \text{at} \ \eta = 0$$

$$f'(\eta) \to 0, \ \theta(\eta) \to 0, \ \phi(\eta) \to 0 \ \text{as} \ \eta \to \infty \quad (15)$$

The important quantities of physical interest are the skin-friction coefficient, $C_f$, Nusselt and Sherwood numbers defined:

$$C_f = \frac{\tau_w}{\rho u_w^2}, \ \text{Nu} = \frac{x q_w}{k(T_w - T_\infty)}, \ \text{Sh} = \frac{x q_m}{D_\phi (C_w - C_x)} \quad (16)$$

where $\tau_w$ is the wall shear stress, $q_w$ – the wall heat flux and $q_m$ – the wall mass flux are given:

$$\tau_w = \mu \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u}{\partial y} \right)_{y=0}, \ q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}, \ q_m = -D_\phi \left( \frac{\partial C}{\partial y} \right)_{y=0} \quad (17)$$

Using the transformed variables (10), the non-dimensional expressions for the $C_f$ skin:

$$C_{f,h}(0) = \left( 1 + \frac{1}{\beta} \right) f''(0)$$

reduced Nusselt number $-\theta'(0)$ and reduced Sherwood number $-\phi'(0)$, respectively defined:

$$C_{h,u}(0) = \sqrt{\frac{2l}{x}} \text{Re}_z C_f, \ -\theta'(0) = \frac{\text{Nu}}{\sqrt{\frac{2l}{x}} \text{Re}_z}, \ -\phi'(0) = \frac{\text{Sh}}{\sqrt{\frac{2l}{x}} \text{Re}_z} \quad (18)$$

where $\text{Re}_z = U_w x / \nu$ is the local Reynolds number based on the stretching velocity. The transformed non-linear ODE (11)-(13) subjected to the boundary conditions (15) are solved numerically by using the Keller-box method.

**Results and discussions**

The coupled non-linear ODE (11)-(13) subjected to the boundary conditions (15) are solved numerically by using the finite difference scheme name as the Keller box method. The numerical results for pertinent flow parameters for Brownian motion parameter, thermophoresis parameter, Casson fluid parameter, Eckert number, chemical reaction parameter, radiation parameter, Prandtl, Lewis, and Hartmann numbers are given in tabular form. Table 1 describes a comparison of the reduced Nusselt number $-\theta'(0)$ with the results given by Bidin and Nazar [12] and Ishak [13]. Table 2 shows the variations of the reduced Nusselt number $-\theta'(0)$, the reduced Sherwood number $-\phi'(0)$ and skin-friction coefficient $C_{f,h}(0)$ for different values of $Nb$, $Nt$, $Pr$, $Le$, $\beta$, $Ec$, $M$, $N$, and $R$. It is noted that the reduced Nusselt number $-\theta'(0)$ decreases for increase in $Nb$, $Ec$, $M$, and $R$ whereas increases for increase in $Nt$, $\beta$, $Pr$, $Le$, and $N$. Where the reduced Sherwood number $-\phi'(0)$ decreases for increase in $Nt$, $\beta$, $M$, and $N$ whereas increases for increase in $Nb$, $Ec$, $Le$, $Pr$, and $R$. Further the skin-friction coefficient $C_{f,h}(0)$ decreases for increase in $Pr$, $M$, $N$, and increases for increase in $\beta$ and $Le$. 
Table 1. Comparison of the reduced Nusselt number $-\theta'(0)$ when $Nb = Nt = Le = R = Ec = 0$ and $\beta \to \infty$

<table>
<thead>
<tr>
<th>Pr</th>
<th>M</th>
<th>N</th>
<th>[12]</th>
<th>[13]</th>
<th>Present results</th>
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<td>5.0</td>
<td>0.8611</td>
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</table>

Table 2. Variations of the local Nusselt number $-\theta'(0)$, the local Sherwood number $-\phi'(0)$ and skin-friction coefficient $C_f(0)$

<table>
<thead>
<tr>
<th>Nb</th>
<th>Nr</th>
<th>$\beta$</th>
<th>Pr</th>
<th>Ec</th>
<th>Le</th>
<th>M</th>
<th>N</th>
<th>R</th>
<th>$-\theta'(0)$</th>
<th>$-\phi'(0)$</th>
<th>$C_f(0)$</th>
</tr>
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<td>0.5</td>
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<td>0.1</td>
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<td>1.2059</td>
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<tr>
<td>0.5</td>
<td>0.1</td>
<td>5.0</td>
<td>6.5</td>
<td>0.5</td>
<td>5.0</td>
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<td>0.1</td>
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<td>5.0</td>
<td>0.1</td>
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<td>0.1</td>
<td>0.9653</td>
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</table>

Figure 2 depicts velocity profile for the different values of $\beta$ by taking fixed values of $Nb, Nr, Le, M, N, R, Pr, Ec$. This behaviour is implicated because of the decreasing yield stress suppressed the velocity field. Figure 3 shows the effects of $M$ on velocity profile $f'(\eta)$ for the fixed values of $Nb, Nr, N, R, Pr, Le, \beta, Ec$. This figure shows that velocity profile $f'(\eta)$ decreases for increasing values of $M$. As $M$ increases, the Lorentz force which opposes the flow, also increases and leads to enhance the deceleration of flow.

It is found from figs. 4 and 5 that temperature profile increase for increasing values of $Nb$ and $Nr$, respectively. In nanofluid motions, particles gain the kinetic energy results for an
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increase in the collisions of particles. Therefore, $\theta(\eta)$ increases for increasing values of $Nb$ and $Nt$, respectively.

Figures 6 shows decrease in concentration profile by increasing Brownian motion parameter $Nb$ whereas increasing values of thermophoresis parameter $Nt$ increases the concentration profile as shown in fig. 7.

Conclusions

Present study numerically investigated the radiation and chemical reaction effects on Casson type MHD nanofluid-flow over an exponentially stretching sheet. Non-Newtonian fluids with the involvement of nanofluid have a great importance due to their superior properties and are beneficial in many fields. Flow phenomenon is characterized by different physical parameters and an analysis is made through graphical and tabulated data. These results can be extended for different flow geometries under different conditions.

It is observed that the reduced Nusselt number $-\theta'(0)$ decreases for increase in $Nb$, Ec, $M$, and $R$ whereas increases for increase in $Nt$, $\beta$, Pr, Le, and $N$. Where the reduced Sherwood number $-\phi'(0)$ decreases for increase in $Nt$, $\beta$, $M$, and $N$ whereas increases for increase in $Nb$, Pr, Ec, Le, and $R$. Further the skin-friction coefficient $C_f(0)$ decreases for increase in Pr, $M$, $N$, and increases for increase in $\beta$ and Le.

References


