HYDROMAGNETIC FALKNER-SKAN FLUID RHEOLOGY WITH HEAT TRANSFER PROPERTIES

by

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This article addresses the effects of heat transfer on magnetohydrodynamic Falkner-Skan wedge flow of a Jeffery fluid. The continuity, momentum and energy balance equations yield the relevant PDE which are transforms to ODE by exploitation of similarity variables. Strength of optimal homotopy series solutions is practiced to solved analytically the transformed ODE model of hydromagnetic Falkner-Skan fluid rheology with heat transfer scenarios. The graphical and numerical illustrations of the result are presented for different interesting flow parameters. Numerical values of Nusselt number are tabulated. It is observed that for the Falkner-Skan rheology, the applied magnetic field acts as a controlling agnet which controls the fluids velocity up to the desired value whereas Debrorah number enhances the fluid velocity.

Key words: Falkner-Skan wedge flow, heat transfer, Jeffery fluid, MHD, OHAM

Introduction

Prandtl [1] introduced the boundary-layer concept for the first time. He described the differences between the viscid and inviscid fluid-flow while presented the investigations using boundary-layer concept. He concluded that the magnitudes of viscous and inertial forces are same near the solid boundaries. Blasius [2] considered the boundary-layer analysis for flow past a flat plate and the results obtained were in an excellent agreement with experimental data. The 2-D boundary-layer analysis for laminar flow was later discussed by Falkner and Skan [3] and analysis is presented by use of the similarity variables. The solution of the reduced ODE was analyzed by Hartree [4]. Thereafter, ample investigations were presented regarding the Falkner-Skan flow under different aspects, few recent studies in this regard are reported in [5-10] and references therein. It is revealed that because of strong non-linearity, the previous studies were often restricted to viscous fluid and provision of numerical solutions only. The numerical solutions are no doubt lacking to predict the real essence for the analysis of the Falkner-Skan fluid flow problem and hence the struggle to derive analytical results with series solutions is still look promising. It is also recognized now that for different industrial fluids, *e.g.*, artificial fibers, paints, molten plastics, blood at low shear rate, food stuffs, shampoo, polymeric liquids, and

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slurries, exhibit rheological characteristics. The consideration of MHD concepts is significant in metallurgy. An important application of MHD flows lies in the distillation of molten materials from non-metallic inclusion through the applied magnetic flux. Keeping in view all these motivated facts, the present article aims to describe the Falkner-Skan wedge flow of Jeffery fluid for the scenarios MHD and heat transfer. The problem statement and solution is included in the next section. Convergence of the series solution is examined [11-20]. Effects of physical parameters on velocity as well as temperature profiles is described. Comparison of our achieved results with already published data is also made. Beside these, there are many reported studies in which the dynamics of MHD nanofluidic problems are investigated in diversified fields, see [21-25] and references cited therein.

Mathematical formulation

Let us consider the 2-D Falkner-Skan wedge flow of a Jeffery fluid in the presence of heat transfer. We further taken into account the analysis of a MHD by exerting the magnetic field in a transverse direction of the flow. The surface temperature is denoted by T_w while the ambient value T_∞ is attained when y tends to infinity. The free stream velocity is denoted by U(x). The small magnitude of magnetic Reynolds number is chosen such that we have negligible induced magnetic field as compared to applied magnetic field. In view of the stated assumptions, boundary layer expression that govern the flow and temperature in dimensional form can be mathematical given as [26]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) = -\frac{\partial p}{\partial x} - \sigma B^{2}(x)u + \left(\frac{v}{1 + \lambda_{1}}\right)\frac{\partial^{2}u}{\partial y^{2}} + \left(\frac{v\lambda_{2}}{1 + \lambda_{1}}\right)\left(\frac{\partial u}{\partial x}\frac{\partial^{2}u}{\partial y\partial x} + u\frac{\partial^{3}u}{\partial y^{2}\partial x} + \frac{\partial v}{\partial y}\frac{\partial^{2}u}{\partial y^{2}} + v\frac{\partial^{3}u}{\partial y^{3}}\right)$$
(2)

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}$$
(3)

with horizontal u and vertical v component of the velocities, ρ – the fluid density, v – the kinematic viscosity, λ_1 – the ratio of relaxation to retardation time, λ_2 – the retardation time, σ – the electrical conductivity, B_0 – the magnetic field strength, k – the thermal conductivity, and c_p – the specific heat.

The relevant boundary conditions can be written as:

$$u = v = 0, \quad T = T_w, \quad y = 0, \quad u \to U(x), \quad T \to T_\infty, \quad \text{for } y \to \infty$$
(4)

where

$$U(x) = ax^{n}, \quad B(x) = B_{0}x^{(n-1)/2}$$
(5)

If ψ is the stream function then by using the following quantities:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \quad \eta = \sqrt{\frac{n+1}{2}} \sqrt{\frac{U}{vx}} y, \quad \psi = \sqrt{\frac{n+1}{2}} \sqrt{vxU} f(\eta), \quad u = Uf'(\eta)$$

$$v = -\sqrt{\frac{n+1}{2}}\sqrt{\frac{\nu U}{x}} \left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta) \right], \quad \theta(\eta) = \frac{T_w - T}{T_w - T_\infty}$$
(6)

Equation (1) is satisfied completely while eqs. (2) to (4) in dimensionless as:

$$f''' + (1 + \lambda_1) \left[ff'' - M^2 (f' - 1) + \frac{2n}{n+1} (1 - f'^2) \right] + \beta \left[(n-1) ff''' + \frac{3n-1}{2} f''^2 - \left(\frac{n+1}{2}\right) ff'''' \right] = 0$$
(7)

$$\theta'' + \Pr f \theta' = 0 \tag{8}$$

$$f(0) = f'(0) = \theta(0) = 0, \quad f'(\infty) = \theta(\infty) = 1$$
 (9)

where primes denotes the differentiation w. r. t η , M is the Hartman number, Pr – the Prandtl number, and β – the Deborah number and are defined:

$$M^{2} = \frac{2\sigma B_{0}^{2}}{\rho a(1+n)}, \quad Pr = \frac{\mu c_{p}}{\alpha}, \quad \beta = \lambda_{2} \frac{U}{x}$$
(10)

While another important physical quantity, *i. e.*, skin friction coefficient, C_f , is formulated by:

$$C_{f} = \frac{1}{\rho[U(x)]^{2}} \tau_{xy}\Big|_{y=0} = \frac{1}{\rho a^{2} x^{2n}} \frac{\mu}{1+\lambda_{1}} \left[\frac{\partial u}{\partial y} + \lambda_{2} \left(u \frac{\partial^{2} u}{\partial x \partial y} + v \frac{\partial^{2} v}{\partial y^{2}} \right) \right]_{y=0}$$
(11)

The dimensionless form of eq. (11) is:

$$C_f = (\operatorname{Re}_x)^{-1/2} \left[f'' + \beta \left(\frac{3n-1}{2} f f'' - \frac{n+1}{2} f f''' \right) \right]_{\eta=0}$$
(12)

where $\operatorname{Re}_{x} = ax^{n+1}(1 + \lambda_{1})/v$ denotes the local Reynolds number.

For Newtonian fluid, system model eqs. (7)-(9) take the form:

$$f''' + ff'' + \frac{2n}{n+1}(1 - f'^2) = 0, \quad \theta'' + \Pr f \theta' = 0$$
(13)

$$f(0) = f'(0) = \theta(0) = 0, \quad f'(\infty) = \theta(\infty) = 1$$
 (14)

Results and disscussion

The graphical and numerical illustration of the results are presented here by using the homotopic series results with in convergence region involves the non-zero auxiliary parameters c_0^f and c_0^{θ} . The optimal values of the parameters c_0^f and c_0^{θ} can be found by using minimal procedure based on mean value of squared residual errors as:

$$E_m^f = \frac{1}{k+1} \sum_{j=0}^k \left\{ N_f \left[\sum_{i=0}^m f(\eta) \right]_{\eta=j\delta\eta} \right\}^2 d\eta$$
(15)

$$E_m^{\theta} = \frac{1}{k+1} \sum_{j=0}^k \left\{ N_{\theta} \left[\sum_{i=0}^m \theta(\eta) \right]_{\eta=j\delta\eta} \right\}^2 d\eta$$
(16)

where $E_m^t = E_m^f + E_m^\theta$ is the sum of the two residual errors. The minimization of the mean residual errors is carried out through built-in procedure of MATHEMATICA package for solving boundary value problems by taking $d\eta = 0.5$ and k = 20 in eqs. (15) and (16). The results

Table 1. Results of optimal convergence onthe basis of residual errors

т	c_0^f	$c_0^{ heta}$	E_m^t	Time
2.0	-1.24	-1.07	5.12×10 ⁻⁴	5.73
4.0	-1.16	-0.99	1.58×10^{-5}	66.48
6.0	-1.02	-0.92	2.05×10^{-5}	5170.48
8.0	-0.97	-0.90	5.82×10^{-7}	6281.19

Table 2. Result of individual residual errors for m = 8 from tab. 1

т	E_m^f	$E_m^{ heta}$	Time
4.0	6.14×10 ⁻⁵	4.81×10 ⁻⁶	14.61
8.0	1.95×10 ⁻⁷	8.67×10 ⁻⁸	100.87
12.0	4.30×10 ⁻⁹	2.53×10 ⁻⁹	245.7



Figure 1. The h- curves for f and θ

of global convergence parameter for different order of the approximation, *i. e.*, m = 2, 4, 6, and 8, are tabulated in tab. 1 along with consume time. Accordingly, in tab. 2, these results are presented for m = 4, 8, and 12. The decreasing trend is seen in the magnitude of residual errors with increases the parameter m, i. e., the order of approximations but at the cost of significant increase in the computing time. The *h*-curves of f and θ are plotted to obtain the reasonable values of h_f and h_{θ} . From fig. 1 it is found that the range of auxiliary parameters is -1.7 $\leq (h_f, h_\theta) \leq -0.5$. Figure 2 is presented just to validate the results. It is found that the residual error in our computation is very small which is nearly equal to zero. In tab. 3, numerical results of skin friction coefficient, $(\text{Re}_x)^{1/2}c_f$, are presented and these results show that with increase in skin friction coefficient by increasing the parameters η , β , and λ_1 , whereas the positive magnitudes of Hartman number results in decrease of the skin friction coefficient.



Figure 2. Residual error in θ

The results for variation in η on f' are presented graphically in fig. 3 which shows that by increasing η , the values of the velocity profile also increases. The description of streamline illustration for the fluids rheology is presented in fig. 3. This plots shows the behavior of momentum boundary-layer near and far away from the wall. Figure 4 plots the effect of β on f'. It is found that effects of β and η are quite similar. The effects of Hartman number and λ_1 are displayed in figs. 5 and 6, respectively. These results show that an increase in the values of Hartman number retards the flow whereas λ_1 causes an increase in the velocity profile. Effects of Hartman number and Prandtl number on temperature profile, θ , are shown in figs. 7 and 8, respectively. It is noticed that the temperature field increases when Hartman number and Prandtl number are increased, while the thickness of thermal boundary-layer decreases by increasing Hartman number and Prandtl number. Table 4 shows the results of $\theta'(\eta)$, various Prandtl numbers, and wedge angle parameters, while tab 5. represents the

1 0 1				
N	β	λ1	М	$(\operatorname{Re}_x)^{1/2}C_f$
0.5	0.2	0.1	0.5	0.85193
1.0	0.2	0.1	0.5	1.14372
1.5	0.2	0.1	0.5	1.32484
0.5	0.0	0.1	0.5	0.84193
0.5	0.2	0.1	0.5	0.89472
0.5	0.4	0.1	0.5	0.92659
0.5	0.2	0.0	0.5	0.17139
0.5	0.2	0.1	0.5	0.25317
0.5	0.2	0.2	0.5	0.31259
0.5	0.2	0.1	0.0	1.16423
0.5	0.2	0.1	0.5	0.91359
0.5	0.2	0.1	1.0	0.31324

 Table 3. Result for skin friction coefficients

 for different physical parameters



Figure 5. Result for variation in M on *f*^{*}

η



Figure 4. Result for variation in β on f'



Figure 6. Result for variation in λ_1 on f'

results for the laminar boundary layer over a wedge. Tables 4 and 5 prove the validity of our work, as our results are in good comparison with already published data.



Figure 7. Result for variation in M on θ

Figure 8. Result for variation in Pr on θ

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1 able 4. Numerical	values of $\theta'(\eta)$	for various Pr and	wedge angel parameters

	$\beta = 0$		$\beta = 0.3$			
Pr	Present study	[27]	[28]	Present study	[27]	[28]
0.1	0.1972	0.1974	0.1980	0.2093	0.2101	0.2090
0.3	0.3047	0.3054	0.3037	0.3285	0.3290	0.3278
0.6	0.3928	0.3923	0.3916	0.4288	0.4290	0.4289
1.0	0.4694	0.4696	0.4696	0.5193	0.5195	0.5195
2.0	0.7976	0.7972	0.5972	0.6688	0.6690	0.6690

f''(0)				
β	Present study	[27]	[29]	
0.0	0.4681	0.4683	0.4696	
0.1	0.5877	0.5879	0.5878	
0.2	0.6873	0.6873	0.6876	
0.4	0.8542	0.8536	0.8549	
0.8	1.1189	1.1188	1.1195	
1.0	1.2311	1.2313	1.2312	

Conclusions

Following are the concluding remarks for the presented study.

- Non-linearity parameter, *n*, enhances the flow. Magnetic field reduces the velocity of fluid.
- By increasing in the intensity of magnetic field, Hartman number decelerates the fluid's velocity and can control the flow field.
- An increase in the magnetic field increases the momentum boundary-layer.
- Retardation time acts as a boosting agent.

- Temperature of the fluid increases with increasing magnetic field.
- The value of the skin friction coefficient increases by increase in the value of the Deborah number.
- Numerical values of Prandtl number and wedge angel parameter are in close approximation with already published work.
- Numerical results for the laminar boundary-layer over a wedge are are in close approximation with already published work.

In future, one may investigate modern soft computing based numerical solvers for the solution of hydromagnetic Falkner-Skan fluidic systems [30-33].

Nomenclature

$B_0 \\ C_{f_f}$	 magnetic field strength skin friction coefficient 	Т и, v	 temperature velocity components
c_0'	 non-zero auxiliary parameter 	х, у	- axes
$\begin{array}{c} c_p \\ c_0^{\theta} \end{array}$	 specific heat at constant pressure non-zero auxiliary parameter 	Greek	symbols
E_m^f	 residual error 	β	 Deborah number
E_m^{t}	 residual error sum 	λ_1	 relaxation time
E_m^{θ}	– residual error	λ_2	 – retardation time
k	- thermal conductivity	V	 kinematic viscosity
М	– Hartman Number	ρ	 – fluid density
Pr	– Prandtl Number	σ	- electrical conductivity
Re_{x}	 Reynolds number 	ψ	– stream function

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