A MODIFICATION OF THE REDUCED DIFFERENTIAL TRANSFORM METHOD FOR FRACTIONAL CALCULUS

by

Kang-Le WANG [a] ***and Kang-Jia WANG*** [b,*]

[a] School of Mathematics and Information Science,
Henan Polytechnic University, Jiaozuo, China
[b] School of Physics and Electronic Information Engineering,
Henan Polytechnic University, Jiaozuo, China

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In this paper, the reduced differential transform method is modified and successfully used to solve the fractional heat transfer equations. The numerical examples show that the new method is efficient, simple, and accurate.

Key words: *Caputo derivative, reduced differential transform, fractional heat transfer equation*

Introduction

In the last few decades, fractional derivatives have found many applications in various fields of physics and engineering, for example, electrical networks, chemical physics, control theory of dynamical systems, reaction diffusion, signal processing, and heat transform can be successfully modeled in linear and non-linear fractional differential equations [1-4]. Various definitions of fractional derivatives are available in open literature [5-9].

Heat transfer is classified into various mechanisms, such as thermal conduction, thermal convection, thermal radiation, and transfer of energy by phase changes. Heat transfer equation is used to describe thermal problems between continuous systems, while its fractional partner can deal with thermal problems in discontinuous medium. In this paper, we consider the following fractional heat transfer equation:

$$\mathrm{D}_t^{\alpha} u - u_{xx} + 2u = 0 \tag{1}$$

with the initial condition:

$$u(x,0) = \mathrm{e}^x$$

where $0 < \alpha \le 1$. When $\alpha = 1$, eq. (1) is the well-known heat transfer equation. The fractional model can be used to describe the transient property of the combination of convective and radiative cooling of a lumped system.

In recent years, many researchers have focused on the approximate analytical solutions of the fractional differential equations and some methods have been developed such as Adomian decomposition method [10], homotopy perturbation method [11-14], variational it-

* Corresponding author, e-mail: konka05@163.com

eration method [15-18], homotopy analysis method [19], Taylor series method [20], Daftardar-Gejji-Jafari method [21], exp-function method [22, 23], and sub-equation method [24], *etc.* In this paper, we have modified the reduced differential transform method (RDTM) [25, 26] for obtaining the approximate analytical solutions of the fractional heat transfer equations.

The modification of the RDTM

In this section, the basic definition of the modified RDTM is introduced.

Denote T the modification of the reduced differential transform of $u(x, t)$ at $t = t_0$ as [25, 26]:

$$T[u(x,t)] = U_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0} \tag{2}$$

where α is a parameter which describes the order of time-fractional derivative.

The differential inverse transform of T at $U_k(x)$ is represented:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)(t-t_0)^{k\alpha} = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=t_0} t^{k\alpha} \tag{3}$$

From eqs. (2) and (3), the following results can be obtained:

If $\quad w(x,t) = u(x,t) \pm v(x,t), \quad$ then $\quad T[w(x,t)] = U_k(x) \pm V_k(x)$

If $\quad w(x,t) = \lambda u(x,t), \quad$ then $\quad T[w(x,t)] = \lambda U_k(x)$

If $\quad w(x,t) = u(x,t)v(x,t), \quad$ then $\quad T[w(x,t)] = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$

If $\quad w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t), \quad$ then $\quad T[w(x,t)] = \frac{(k+r)!}{k!}\frac{\partial^r}{\partial t^r}U_{k+r}(x)$

If $\quad w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t), \quad$ then $\quad T[w(x,t)] = \frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x)$

If $\quad w(x,t) = [u(x,t)]^k, \quad$ then $\quad T[w(x,t)] = W_k(x) = \begin{cases} U_0(x), & k=0 \\ \sum_{n=1}^{k}\frac{(m+1)n-k}{kU_0(x)}U_n(x)W_{k-n}(x), & k \geq 1 \end{cases}$

By using the modification of the reduced differential transform on the both sides of eq. (1) and the initial condition, we have the following form:

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x) = \frac{\partial^2}{\partial x^2}U_k(x) - 2U_k(x) \tag{4}$$

$$U_0(x) = \mathrm{e}^x \tag{5}$$

By iterative calculation on eqs. (4) and (5), we have:

$$U_1(x) = -\frac{e^x}{\Gamma(\alpha+1)}, \quad U_2(x) = \frac{e^x}{\Gamma(2\alpha+1)}, \quad U_3(x) = -\frac{e^x}{\Gamma(3\alpha+1)}, \quad U_4(x) = \frac{e^x}{\Gamma(4\alpha+1)},$$

$$U_5(x) = -\frac{e^x}{\Gamma(5\alpha+1)}, \quad U_6(x) = \frac{e^x}{\Gamma(6\alpha+1)},$$

$$\cdots$$

$$U_n(x) = (-1)^n \frac{e^x}{\Gamma(n\alpha+1)}$$

So we have the following solution:

$$u(x,t) = U_0(x)t^0 + U_1(x)t^\alpha + U_2(x)t^{2\alpha} + U_3(x)t^{3\alpha} + U_4(x)t^{4\alpha} + \cdots$$

$$= e^x t^0 - \frac{e^x}{\Gamma(\alpha+1)}t^\alpha + \frac{e^x}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{e^x}{\Gamma(3\alpha+1)}t^{3\alpha} + \frac{e^x}{\Gamma(4\alpha+1)}t^{4\alpha} - \cdots \tag{6}$$

When $\alpha = 1$, the exact solution of eq. (1) is given:

$$u(x,t) = e^{x-t} \tag{7}$$

Equation (6) tends to the exact solution when $\alpha = 1$.

In tab. 1, we compare the exact solution with the 10^{th} order approximate solution for different values of α. By comparison, it is easy to find that the approximate solutions continuously depend on the values of time-fractional derivative. The transient property of the combination of convective and radiative cooling of a lumped system is determined by the value of α.

Table 1. Comparison between the exact solution and the 10^{th} order approximate solution by modified RDTM for different values of α

x	t	α			$u_{\text{exa}}(\alpha = 1)$
		0.6	0.8	1	
0.2	0.3	0.7538699863	0.8255259277	0.9048374178	0.9048374180
0.4	0.6	0.7483607341	0.7721212773	0.8187307534	0.8187307531
0.5	0.7	0.7830989649	0.7890770719	0.8187307542	0.8187307531
0.8	0.9	0.9605377311	0.9213737964	0.9048374342	0.9048374180
0.3	0.5	0.7193903227	0.7590522835	0.8187307535	0.8187307531
0.7	0.3	1.242921482	1.361062156	1.4918246970	1.4918246980
0.8	0.6	1.116423025	1.151869590	1.2214027580	1.2214027580
0.5	0.9	0.7115838531	0.6825704967	0.6703200583	0.6703200460

Conclusion

In this paper, we have successfully modified the RDTM. We use the method to find the approximate analytical solutions of fractional heat transfer equation. Our results show that the modification of the reduced differential transform is more reliable, efficient and accurate.

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