NEW APPROACH TO PARAMETERIZED HOMOTOPY PERTURBATION METHOD

by

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In this paper, new approach to parameterized homotopy perturbation method is presented to solve non-oscillatory problems. In contrast to the classical version of the homotopy method, optimal value of α is identified and used to obtain approximate solutions. The new approach is powerful as it effectively handled non-oscillatory problems and gives results with the smallest known errors.

Key words: *parameterized homotopy perturbation and non-oscillatory equations*

Introduction

Homotopy perturbation method is one of the most recent and powerful methods developed to tackle both linear and non-linear problems. A lot of analytical numerical methods have been used to solve linear and non-linear problems. These include parameterized homotopy perturbation method (PHPM) [1], frequency-amplitude formulation method [2-4], energy balance method [5-7], variational iteration method [8, 9], homotopy perturbation method [10-13], and parameter expanding method [14-16]. In a recent paper [1], Duffing equation which is a well-known oscillatory problem [17], was used to elucidate on the effectiveness and the applicability of the PHPM.

In the solution procedure of PHPM to oscillatory problems, the homotopy is constructed such that $\cos\omega t$ is introduced on both sides of the equation to cancel out the presence of the zero term on the right hand side [1]. This helps to easily manipulate the transformed problem so that the exact angular frequency, ω_{ex}, can be used to identify the optimal value of α. However, in this paper a new approach to PHPM is presented in order to solve non-oscillatory problems.

Basic idea of the new approach

To illustrate the basic idea of the new approach for solving non-oscillatory problems, we consider the following non-oscillatory equation of the form:

$$\varphi(u) - f(t) = 0, \qquad t \in R^d \tag{1}$$

with boundary conditions:

$$u(0) = A \in R, \qquad u'(0) = B \in R \tag{2}$$

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where φ is a general differential operator, A and B are the known boundary conditions, and $f(t)$ – a known analytical function.

The operator φ can generally be divided into two parts, linear, L, and non-linear, N. Equation (1), can therefore, be re-written:

$$L(u) + N(u) - f(t) = 0 \tag{3}$$

We construct the following homotopy:

$$H(u,p) = (1-\alpha p)\left[L(u) - L(u_0)\right] + \alpha p\left[\varphi(u) - f(t) - 1\right] = p\left(\frac{1}{p^{\alpha p}} - 2\alpha\right) \tag{4}$$

or

$$H(u,p) = L(u) - L(u_0) + \alpha p(u_0) + \alpha p\left[N(u) - f(t) - 1\right] = p\left(\frac{1}{p^{\alpha p}} - 2\alpha\right), \quad p \in \left[0, \frac{1}{\alpha}\right] \tag{5}$$

where $p = 1/\alpha$ is the embedding parameter, u_0 – the initial approximate solution of eq. (1) which satisfies the boundary conditions eq. (2).

Obviously:

$$H(u,0) = L(u) - L(u_0) = 0 \tag{6}$$

$$H\left(u, \frac{1}{\alpha}\right) = \varphi(u) - f(t) = 0 \tag{7}$$

Now we use p as expanding parameter and assume the solution of eq. (4) or eq. (5) can be written in the form:

$$u = u_0 + pu_1 \tag{8}$$

Now setting $p = 1/\alpha$, as in [1] the approximate solution of eq. (3) becomes:

$$u = \lim_{p \to \frac{1}{\alpha}} u = u_0 + \frac{1}{\alpha} u_1 \tag{9}$$

Applications/results

Example 1

We consider a governing equation of cooling problem with its initial conditions as in [18]:

$$\rho V_c \frac{\mathrm{d}T}{\mathrm{d}t} + hA\left(T - T_a\right) + E\delta A(T^4 - T_s^4) = 0, \qquad t = 0, \quad \tau = T_i \tag{10}$$

To solve eq. (10), using the idea of PHPM we adopt its simplified version as given in [18] in the form:

$$\frac{d\theta}{d\tau}+\theta+\varepsilon\theta^4=0, \quad \tau=0, \quad \theta=1 \tag{11}$$

we construct the following homotopy:

$$\frac{d\theta}{d\tau}+\alpha p\left[\theta+\varepsilon\theta^4-1\right]=p\left(\frac{1}{p^{\alpha p}}-2\alpha\right), \qquad p\in\left[0,\frac{1}{\alpha}\right] \tag{12}$$

here $p\in[0,1/\alpha]$ is the embedding parameter. At $p = 0$, eq. (12) becomes linear and at $p=1/\alpha$ eq. (12) turns to the original problem.

Using the parameter p, we expand the solution $\theta(\tau)$ as follows:

$$\theta=\theta_0+p\theta_1 \tag{13}$$

Setting $p = 1/\alpha$ leads to the approximate solution of eq. (11):

$$\theta=\lim_{p\to 1/\alpha}\theta=\theta_0+\frac{1}{\alpha}\theta_1 \tag{14}$$

Substituting eq. (13) into eq. (12) and equating likes powers of p, we have:

$$p^0:\theta_0=0, \qquad \tau=0, \qquad \theta_0=1 \tag{15}$$

$$p^1:\theta_1+\alpha[\theta+\varepsilon\theta_0^4-1]=\left(\frac{1}{p^{\alpha p}}-2\alpha\right), \qquad \tau=0, \qquad \theta_0=1 \tag{16}$$

Solving eqs. (15) and (16) we have:

$$\theta_0=1 \tag{17}$$

$$\theta_1=\left(\frac{1}{p^{\alpha p}}-\alpha\right)\tau-\alpha(1+\varepsilon)\tau \tag{18}$$

Now putting eq. (17) and eq. (18) into eq. (14), we have:

$$\theta=1+\left[\left(\frac{1}{\alpha p^{\alpha p}}-1\right)-(1+\varepsilon)\right]\tau \tag{19}$$

Putting $p^{\alpha p} = 1$, we can rewrite eq. (19) in the form:

$$\theta_p=1+\left[\frac{1}{\alpha}-(2+\varepsilon)\right]\tau \tag{20}$$

The exact solution of eq. (11) was given by [18]:

$$\tau=\frac{1}{3}\ln\frac{1+\varepsilon\theta^3}{(1+\varepsilon)\theta^3} \tag{21}$$

In order to identify the optimal alpha we use eq. (21):

$$\frac{1}{\alpha} = (2+\varepsilon) + \frac{1}{\tau}\left[\sqrt[3]{\frac{\mathrm{e}^{-3\tau}}{\left[(1+\varepsilon) - \varepsilon \mathrm{e}^{-3\tau}\right]}} - 1\right] \tag{22}$$

Table 1. Numerical results of eq. (11) for $\varepsilon = 1$

τ	α	θ_{ex}	θ_p	$\lvert\theta_{\mathrm{ex}} - \theta_p\rvert$
0.1	0.72500542	0.837930003	0.837930003	$3.37 \cdot 10^{-10}$
0.2	0.618892372	0.723157965	0.723157965	$8.25 \cdot 10^{-10}$
0.3	0.561524396	0.634259957	0.634259957	$6.03 \cdot 10^{-10}$
0.4	0.525084774	0.561781752	0.561781753	$1.74 \cdot 10^{-9}$
0.5	0.499617646	0.500765292	0.500765293	$1.23 \cdot 10^{-9}$

Example 2

Consider the following third-order linear differential equation with three point boundary conditions:

$$u'''(x) - k^2 u'(x) + a = 0, \qquad 0 \le x \le 1 \tag{23}$$

with conditions:

$$u'(0) = u'(1) = 0, \qquad u(0.5) = 0 \tag{24}$$

The exact solution of eq. (23) was given in [19]:

$$u(x) = \frac{a}{k^3}\left(\sinh\frac{k}{2} - \sinh kx\right) - \frac{a}{k^2}\left(x - \frac{1}{2}\right) + \frac{a}{k^3}\tanh\frac{k}{2}\left(\cosh kx - \cosh\frac{k}{2}\right) \tag{25}$$

where $k = 5$ and $a = 1$.

Using the PHPM, we construct the following homotopy:

$$u''' - \alpha p\left[25u' + 1\right] + 1 = p\left(\frac{1}{p^{\alpha p}} - 2\alpha\right), \qquad p \in \left[0, \frac{1}{\alpha}\right] \tag{26}$$

where $p \in [0, 1/\alpha]$ is the embedding parameter. Assume the solution of eq. (23) is:

$$u = u_0 + pu_1 \tag{27}$$

setting $p = 1/\alpha$, we have:

$$u = \lim_{p \to \frac{1}{\alpha}} u = u_0 + \frac{1}{\alpha}u_1 \tag{28}$$

Putting eq. (27) into eq. (26) and equating the identical powers of p gives:

$$p^0: u_0''' + 1 = 0, \qquad u_0'(0) = 0, \qquad u_0(0) = A, \qquad u_0''(0) = B \tag{29}$$

$$p^1: u_1''' - \alpha 25 u_0' - \alpha = \left(\frac{1}{p^{\alpha p}} - 2\alpha\right), \qquad u_2'(0) = 0, \qquad u_2(0) = 0, \qquad u_2''(0) = 0 \tag{30}$$

Solving eqs. (29) and (30) we have:

$$u_0 = A + \frac{Bx^2}{2} - \frac{x^3}{6} \tag{31}$$

$$u_1 = \left(\frac{1}{p^{\alpha p}} - \alpha\right)\frac{x^3}{6} + \alpha\left(\frac{25Bx^4}{24} - \frac{5x^5}{24}\right) \tag{32}$$

[20] gives:

$$A = -0.012107085822126442$$

$$B = 0.19732286064025403$$

Now putting eqs. (31) and (32) into eq. (28):

$$u = A + \frac{Bx^2}{2} - \frac{x^3}{6} + \frac{5}{24}(5Bx^4 - x^5) + \left(\frac{1}{\alpha p^{\alpha p}} - 1\right)\frac{x^3}{6} \tag{33}$$

Taking $p^{\alpha p} = 1$ we have:

$$u_p = A + \frac{Bx^2}{2} - \frac{x^3}{6} + \frac{5}{24}(5Bx^4 - x^5) + \left(\frac{1}{\alpha} - 1\right)\frac{x^3}{6} \tag{34}$$

The parameter α can be identified optimally in the form:

$$\frac{1}{\alpha} = 1 + \frac{6}{x^3}\left[-A - \frac{Bx^2}{2} + \frac{x^3}{6} - \frac{5}{24}(5Bx^4 - x^5) + \frac{a}{k^3}\left(\sinh\frac{k}{2} - \sinh kx\right) + \right.$$
$$\left. + \frac{a}{k^2}\left(x - \frac{1}{2}\right) + \frac{a}{k^3}\tanh\frac{k}{2}\left(\cosh kx - \cosh\frac{k}{2}\right)\right] \tag{35}$$

Table 2. Numerical results of eq. (23)

u	α	u_{ex}	u_p	$\lvert u_{ex} - u_p \rvert$
0.1	0.999040607	–0.011268507	–0.011268507	$3.23 \cdot 10^{-14}$
0.2	0.992887944	–0.009222206508	–0.009222206507	$5.40 \cdot 10^{-13}$
0.3	0.977829921	–0.00646686151	–0.006466868148	$3.31 \cdot 10^{-12}$
0.4	0.951889256	–0.003320195353	–0.003320195353	$3.30 \cdot 10^{-14}$
0.5	0.914855315	0	$1.590325 \cdot 10^{-11}$	$1.59 \cdot 10^{-11}$

Conclusion

Homotopy of PHPM has been successfully constructed to handle non-oscillatory problems. The new approach is powerful and effective. Solutions obtained are highly accurate and has the best and smallest known errors.

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