LATTICE BOLTZMANN MODEL FOR THE RIESZ SPACE FRACTIONAL REACTION-DIFFUSION

by

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In this paper, a Riesz space fractional reaction-diffusion equation with non-linear source term is considered on a finite domain. This equation is commonly used to describe anomalous diffusion in thermal science. To solve the diffusion equation, a new fractional lattice Boltzmann method is proposed. Firstly, a difference approximation for the global spatial correlation of Riesz fractional derivative is derived by applying the numerical discretization technique, and a brief convergence analysis is presented. Then the global spatial correlation process is inserted into the evolution process of the standard lattice Boltzmann method. With combining Taylor expansion, Chapman-Enskog expansion and the multi-scales expansion, the governing evolution equation is recovered from the continuous Boltzmann equation. Three numerical examples are provided to confirm our theoretical analysis and illustrate the effectiveness of our method at last.

Key words: *lattice Boltzmann method, non-linear source term, Taylor expansion, Riesz space fractional reaction-diffusion equation, multi-scales expansion*

Introduction

In 1855, Fick first proposed the diffusion equation to describe transport phenomena of nutrients in living organism. The important equation he utilized is the famous Fick's first law, $J = -k\nabla u$ where J is the flux of the fluid-flow, k – the diffusion coefficient, and u – the concentration. Diffusion under Fick's law is determined by the local gradient of concentration. In recent years more and more diffusion phenomena are found to be non-Fickian, which can be described by fractional operators. A generalized Fick's law was presented in [1]:

$$J = -k_{\mu}\left(\omega_1\,{}^{+}\mathrm{D}_x^{\mu} + \omega_2\,{}^{-}\mathrm{D}_x^{\mu}\right)u \tag{1}$$

where ${}^{+}\mathrm{D}_x^{\mu}$ and ${}^{-}\mathrm{D}_x^{\mu}$ represent left and right fractional Riemann-Liouville (R-L) derivatives respectively. Applying continuity equation with a source or sink term $f(x, t)$, we arrive at the following space-fractional reaction diffusion equation:

$$\frac{\partial u}{\partial t} = \nabla\left[k_{\mu}\left(\omega_1\,{}^{+}\mathrm{D}_x^{\mu} + \omega_2\,{}^{-}\mathrm{D}_x^{\mu}\right)u\right] + f(x,t) \tag{2}$$

For a fixed k_{μ} which does not change with x, eq. (2) reduces to:

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$$\frac{\partial u}{\partial t} = k_\mu \left(\omega_1 \, {}^{+}\mathrm{D}_x^\alpha + \omega_2 \, {}^{-}\mathrm{D}_x^\alpha \right) u + f(x,t) \tag{3}$$

where $\alpha = 1 + \mu$. If specifying:

$$\omega_1 = \omega_2 = -\frac{1}{2\cos\frac{\alpha\pi}{2}}$$

we will derive the Riesz fractional reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = k_\mu \frac{\partial^\alpha u}{\partial |x|^\alpha} + f(x,t) \tag{4}$$

and the Riesz fractional derivative is defined:

$$\frac{\partial^\alpha u}{\partial |x|^\alpha} = \frac{-1}{2\cos\frac{\alpha\pi}{2}\Gamma(2-\alpha)} \frac{\partial^\alpha}{\partial x^2} \int_{-\infty}^{+\infty} |x-\xi|^{1-\alpha} u(\xi)\mathrm{d}\xi, \qquad 1<\alpha<2 \tag{5}$$

Equation (3) was first proposed by Chaves [1] to investigate the mechanism of super-diffusion and was later generalized by Benson *et al.* [2, 3]. It is a powerful approach to description of transport dynamics in complex systems governed by anomalous diffusion. Many numerical approaches have been proposed for eq. (3). Tadjeran [4] proposed a shifted Grunwald formula for R-L space fractional derivative and obtained an unconditional stable second-order accurate numerical approximation by applying Crank-Nicholson technique in time and extrapolation in space. He and Li [5] suggested a transform to convert fractional differential equations into PDE. Zhang *et al.* [6] considered eq. (4) with Galerkin finite element method in space and a backward difference technique in time, both the stability and convergence were proven. Sousa [7] derived a second order numerical method for eq. (3) involving a convection term and studied its convergence. Xu *et al.* [8] proposed a discontinuous Galerkin method for eq. (4) and derived stability analysis and optimal convergence rate.

On the other hand, as a kind of mesoscopic numerical simulation method, lattice Boltzmann method (LBM) is a mature and popular approach to deal with fluid mechanics in the areas of a mesoscopic calculation [9-12]. In recent twenty years, many scholars proposed LBM for a series of classical PDE, such as the reaction-diffusion equations [13], the convection-diffusion equation [14-16], Klein-Gordon equation [17], Sine-Cosine-Gordon equation [18], Burgers' equation [19], and Korteweg-de Vries equation [20], *etc.* However, to the best of our knowledge, there are few results about the applications of LBM in fractional equations. Recently, Zhang and Yan [21] proposed LBM for the time fractional-order sub-diffusion equation based on the definition of R-L fractional derivative. Zhou *et al.* [22] employed LBM for solving the space fractional-order advection-diffusion equation with a sink term $f(x, t)$.

As in classical diffusion equations, f is usually related to the concentration u which is our original motivation to study eq. (4). Based on the previous observations, in this paper, we consider the following Riesz space fractional reaction-diffusion equation (RSFRDE) with non-linear source term:

$$\frac{\partial u}{\partial t} = k_\mu \frac{\partial^\alpha u}{\partial |x|^\alpha} + f(u,x,t), \qquad 0 < x < L, \qquad 0 < t \le T \tag{6}$$

subject to the initial condition as $u(x, 0) = \psi(x)$, $0 < x < L$ and the boundary condition as $u(0, t) = \varphi_0(t)$, $u(L, t) = \varphi_L(t)$, $0 < x \le T$ where $k_\mu > 0$ represents the dispersion coefficient. Gorenflo and Mainaedi [23] defined the Riesz space fractional operator on a finite domain $a \le x \le b$:

$$\frac{\partial^\alpha u}{\partial |x|^\alpha} = \frac{-1}{2\cos\frac{\alpha\pi}{2}}\left[\mathrm{D}_{a+}^\alpha f(x) + \mathrm{D}_{b-}^\alpha f(x)\right], \qquad 1 < \alpha < 2 \tag{7}$$

where

$$\mathrm{D}_{a+}^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_a^x \frac{f(\xi)\mathrm{d}\xi}{(x-\xi)^{\alpha-1}} \text{ and } \mathrm{D}_{b-}^\alpha f(x) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^b \frac{f(\xi)\mathrm{d}\xi}{(\xi-x)^{\alpha-1}} \text{ and } \Gamma(\cdot)$$

represents the Euler gamma function. Based on finite difference method and standard LBM, we develope a new space fractional lattice Boltzmann method (FLBM) for the simulation of fractional PDE.

Numerical discretization techniques of Riesz space-fractional derivatives

In order to make the standard LBM suitable for RSFRDE, eq. (6), in this section, we first deal with the numerical discretization techniques of the Riesz space-fractional derivative, and then present the error.

Finite difference approximation

We now sketch a redistribution scheme and a related discrete random walk model for eq. (6). The case of standard diffusion process, *i. e.* $\alpha = 2$ is included, which actually leads to a discrete model of the classical Brownian motion.

For our purpose we discretize space and time by grid points and time instants. For convenience sake, we denote the weight coefficient functions by:

$$\omega(j,\alpha) = (j+1)^{2-\alpha} - j^{2-\alpha} \tag{8}$$

where α is the order of fractional derivatives, and j is the distance between x and grid points. The functions $\omega(j, \alpha)$ are weight factors of the equilibrium distribution functions, and they are derived theoretically and non-negative.

Assume that the spatial domain is $[a, b]$, then let $a = x_0 < x_1 < \ldots < x_{L-1} < x_L = b$ denote the nodal points with equispace $h = (b - a)/L$. For notation convenience, we denote:

$$\Psi_1(x,t) := \int_a^x \frac{u(\xi,t)\mathrm{d}\xi}{(x-\xi)^{\alpha-1}}, \qquad \Psi_2(x,t) := \int_x^b \frac{u(\xi,t)\mathrm{d}\xi}{(\xi-x)^{\alpha-1}}, \quad \Psi(x,t) := \Psi_1(x,t) + \Psi_2(x,t) \tag{9}$$

The function $\Psi_1(x, t)$ at point $x = x_l$ can be approximated by:

$$\Psi_1(x,t)=\int_a^x \frac{u(\xi,t)\mathrm{d}\xi}{(x-\xi)^{\alpha-1}}=\int_0^{x-a}\frac{u(x-\xi,t)\mathrm{d}\xi}{\xi^{\alpha-1}}=\sum_{j=0}^{l-1}\int_{jh}^{(j+1)h}\frac{u(x-\xi,t)\mathrm{d}\xi}{\xi^{\alpha-1}}\approx$$

$$\approx\frac{1}{2}\sum_{j=0}^{l-1}\left\{u(x-jh,t)+u\left[x-(j+1)h,t\right]\right\}\sum_{j=0}^{l-1}\int_{jh}^{(j+1)h}\frac{\mathrm{d}\xi}{\xi^{\alpha-1}}=$$

$$=\frac{h^{2-\alpha}}{4-2\alpha}\sum_{j=0}^{l-1}\omega(j,\alpha)\left\{u(x-jh,t)+u\left[x-(j+1)h,t\right]\right\} \tag{10}$$

Similarly, the function $\Psi_2(x, t)$ can be approximated by:

$$\Psi_2(x,t)=\frac{h^{2-\alpha}}{4-2\alpha}\sum_{k=0}^{L-l-1}\omega(k,\alpha)\left\{u(x+kh,t)+u\left[x+(k+1)h,t\right]\right\} \tag{11}$$

The preceding two approximations lead to:

$$\Psi(x,t)=\Psi_1(x,t)+\Psi_2(x,t)=$$

$$=\frac{h^{2-\alpha}}{4-2\alpha}\left(\sum_{j=0}^{l-1}\omega(j,\alpha)\{u(x-jh,t)+u[x-(j+1)h,t]\}+\right.$$

$$\left.+\sum_{k=0}^{L-l-1}\omega(k,\alpha)\{u(x+kh,t)+u[x+(k+1)h,t]\}\right) \tag{12}$$

The following result is obtained by inserting eq. (12) into eq. (6). Thus, eq. (6) can be approximated with:

$$\frac{\partial u}{\partial t}=\lambda\frac{\partial^2\tilde{u}(x,t)}{\partial x^2}+f(u,x,t) \tag{13}$$

where $\tilde{u}\left(x,t\right)$ and λ are the following expressions, respectively:

$$\tilde{u}(x,t)=\left\{\sum_{j=0}^{l-1}w(j,\alpha)u(x-jh,t)+u\left[x-(j+1)h,t\right]\right\}+$$

$$+\left\{\sum_{k=0}^{L-l-1}w(k,\alpha)u(x+kh,t)+u\left[x+(k+1)h,t\right]\right\} \tag{14}$$

$$\lambda=-\frac{k_u h^{2-\alpha}}{4\Gamma(3-\alpha)\cos\dfrac{\alpha\pi}{2}} \tag{15}$$

Analysis of error

Next we present a brief analysis of the error from eq. (6) to eq. (13).

Theorem 1. Let $1<\alpha\le 2$ If $f'(x)\in C^1[a,b]$, then:

$$\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}(x_n-x)^{1-\alpha}\left[f(x)-\frac{f(x_j)+f(x_{j-1})}{2}\right]\mathrm{d}x\le C_f h^{3-\alpha} \tag{16}$$

where C_f is a constant only dependent on the function f.

Proof. Denote:

$$A=\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}(x_n-x)^{1-\alpha}\left[f(x)-\frac{f(x_j)+f(x_{j-1})}{2}\right]\mathrm{d}x$$

The functions $f(x_j)$ and $f(x_{j-1})$ are expanded using Taylor series at x:

$$f(x_j)=f(x)+f'(x)(x_j-x)+O(h^2) \tag{17}$$

and

$$f(x_j-1)=f(x)+f'(x)(x_{j-1}-x)+O(h^2) \tag{18}$$

then $$A=\sum_{j=1}^{n}\int_{x_{j-i}}^{x_j}(x_n-x)^{1-\alpha}\mathrm{d}x+O(h^2)\le C_f\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}(x_n-x)^{1-\alpha}(x_j+x_{j-1}-2x)\mathrm{d}x+O(h^2) \tag{19}$$

From Lin and Xu [24] we know:

$$\int_{x_{j-1}}^{x_j}(x_n-x)^{1-\alpha}(x_j+x_{j-1}-2x)\mathrm{d}x\le Ch^{3-\alpha} \tag{20}$$

hence

$$A=\sum_{j=1}^{n}\int_{x_{j-i}}^{x_j}(x_n-x)^{1-\alpha}\frac{f'(x)(x_j+x_{j-1}-2x)}{2}\mathrm{d}x+O(h^2)\le C_f h^{3-\alpha} \tag{21}$$

The LBM for the RSFRDE

The establishment of the FLBM

It is generally known that the collision and streaming processes of the standard LBM are usually expressed by the following equations:

$$\textit{Collision:}\quad f_i^*(x,t)=f_i(x,t)-\frac{1}{\tau}\left[f_i(x,t)-f_i^{\mathrm{eq}}(x,t)\right] \tag{22}$$

and

$$\textit{Streaming:}\ f_i(x+\Delta tc_i,t+\Delta t)=f_i^*(x,t) \tag{23}$$

from which we can derive:

$$f_i(x+\Delta tc_i,t+\Delta t)=f_i(x,t)-\frac{1}{\tau}\left[f_i(x,t)-f_i^{\mathrm{eq}}(x,t)\right] \tag{24}$$

Collision and streaming processes are only related to their local equilibrium values in standard LBM, but the RSFRDE has global spatial correlation. So, based on the standard LBM, FLBM model can be put in the following steps by adding process of the global spatial correlation.

– *Step 1. The evolution process:*

$$f_i(x+\Delta t c_i, t+\Delta t) = f_i(x,t) - \frac{1}{\tau}\left[f_i(x,t) - f_i^{\text{eq}}(x,t)\right] + \Delta t\theta_i(x,t) + \frac{\Delta t^2}{2}\frac{\partial \theta_i(x,t)}{\partial t} \tag{25}$$

where $\Delta t\theta_i(x,t) + \dfrac{\Delta t^2}{2}\dfrac{\partial \theta_i(x,t)}{\partial t}$ is given to recover the non-linear source term.

– *Step 2. The macroscopic quantity:*

$$\rho(x,t+\Delta t) = \sum_i f_i(x,t+\Delta t) \tag{26}$$

– *Step 3. The global spatial correlation process:*

$$u(x,t) = \sum_j \omega(j,\alpha)\rho(x+jh,t) + \sum_k \omega(k,\alpha)\rho(x-kh,t) \tag{27}$$

where ω is defined by eq. (8).

Recovery of the RSFRDE

In order to recover the macroscopic equation, we employ Chapman-Enskog expansion [25] to $f_j(x, t)$ under the assumption of small Knudsen number, ε, then discuss the changes in different time scales. Denote t_0, t_1 as the time scales, then we have:

$$f_i = f_i^{(0)} + \varepsilon f_i^{(1)} + \varepsilon^2 f_i^{(2)} + O(\varepsilon^3), \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\frac{\partial}{\partial t_1} + O(\varepsilon^2) \tag{28}$$

For fractional-order diffusion problem, by the law of conservation, it is appropriate to assume that equilibrium distribution functions are constants, where no macroscopic velocity is involved. Equilibrium distribution function satisfies the constraint of mass and momentum:

$$\sum_i f_i^{\text{eq}} = \sum_i f_i^{(0)} = u, \qquad \sum_i c_i f_i^{\text{eq}} = 0 \tag{29}$$

combining eqs. (28), we obtain:

$$\sum_i f_i^{(1)} = \sum_i f_i^{(2)} = 0 \tag{30}$$

Using Taylor expansion on eq. (25) at point (x, t), with setting $\varepsilon = \Delta t$, we have:

$$\varepsilon\left(c_i\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) f_i + \frac{1}{2}\varepsilon^2\left(c_i\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^2 f_i + O(\varepsilon^2) = \frac{1}{\tau}\left(f_i - f_i^{\text{eq}}\right) + \varepsilon^2\theta_i + \frac{\varepsilon^3}{2}\frac{\partial\theta_i}{\partial t} \tag{31}$$

Substituting eqs. (28) into eq. (33), we get:

$$\varepsilon\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}\right)\left(f_i^{\text{eq}}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}\right)+$$

$$+\frac{1}{2}\varepsilon^2\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}\right)^2\left(f_i^{\text{eq}}+\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}\right)=$$

$$=-\frac{1}{\tau}\left(\varepsilon f_i^{(1)}+\varepsilon^2 f_i^{(2)}\right)+\varepsilon^2\theta_i+\frac{\varepsilon^3}{2}\frac{\partial\theta_i}{\partial t} \tag{32}$$

Equating term from both sides of the previous equation of same order of ε.

– Terms order of ε:

$$\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}\right)f_i^{\text{eq}}=-\frac{1}{\tau}f_i^{(1)} \tag{33}$$

– Terms order of ε^2:

$$\frac{\partial f_i^{\text{eq}}}{\partial t_1}+\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}\right)f_i^{(1)}+\frac{1}{2}\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}\right)^2 f_i^{\text{eq}}=-\frac{1}{\tau}f_i^{(2)}+\theta_i+\frac{\varepsilon}{2}\frac{\partial\theta_i}{\partial t} \tag{34}$$

Substituting $f_i^{(1)}$ in eq. (33) into eq. (34), yields:

$$\frac{\partial f_i^{\text{eq}}}{\partial t_1}+\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}\right)\left[\left(\frac{1}{2}-\tau\right)\left(c_i\frac{\partial}{\partial x}+\frac{\partial}{\partial t_0}\right)f_i^{\text{eq}}\right]=-\frac{1}{\tau}f_i^{(2)}+\theta_i+\frac{\varepsilon}{2}\frac{\partial\theta_i}{\partial t} \tag{35}$$

To recover continuum eq. (6), summing up i in eqs. (33) and (34) over all states, respectively, namely, from $i = 1$ to $i = 3$ we arrive at:

$$\frac{\partial}{\partial x}\left(\sum_i c_i f_i^{\text{eq}}\right)+\frac{\partial}{\partial t_0}\left(\sum_i f_i^{\text{eq}}\right)=-\frac{1}{\tau}\sum_i f_i^{(1)} \tag{36}$$

and

$$\frac{\partial}{\partial t_1}\sum_i f_i^{\text{eq}}+\left(\frac{1}{2}-\tau\right)\left(\frac{\partial}{\partial x^2}\sum_i c_i^2 f_i^{\text{eq}}+2\frac{\partial^2}{\partial_{t_0}\partial x}\sum_i c_i f_i^{\text{eq}}+\frac{\partial^2}{\partial t_0^2}\sum_i f_i^{\text{eq}}\right)=$$

$$=-\frac{1}{\tau}\sum_i f_i^{(2)}+\sum_i\theta_i+\frac{\varepsilon}{2}\frac{\partial\sum_i\theta_i}{\partial t} \tag{37}$$

Operating (37) × ε + (36), we have:

$$\frac{\partial}{\partial x}\left[\sum_i\left(c_i f_i^{\text{eq}}\right)\right]+\left(\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}\right)\left(\sum_i f_i^{\text{eq}}\right)+\frac{\partial^2}{\partial x^2}\left[\left(\frac{1}{2}-\tau\right)\varepsilon\left(\sum_i c_i c_i f_i^{\text{eq}}\right)\right]+$$

$$+\varepsilon\frac{\partial}{\partial x}\left\{\left(\frac{1}{2}-\tau\right)\left[\frac{\partial}{\partial t_0}\left(\sum_i c_i f_i^{\text{eq}}\right)\right]\right\}=\sum_i \varepsilon\theta_i+\frac{\varepsilon}{2}\frac{\partial\sum_i\theta_i}{\partial t} \tag{38}$$

Back to the original time scale:

$$\frac{\partial}{\partial x}\left[\sum_i\left(c_i f_i^{\text{eq}}\right)\right]+\frac{\partial}{\partial t}\left(\sum_i f_i^{\text{eq}}\right)+\varepsilon\left(\frac{1}{2}-\tau\right)\frac{\partial^2}{\partial x^2}\left(\sum_i c_i c_i f_i^{\text{eq}}\right)+$$
$$+\varepsilon\frac{\partial}{\partial x}\left\{\left(\frac{1}{2}-\tau\right)\left[\frac{\partial}{\partial t_0}\left(\sum_i c_i f_i^{\text{eq}}\right)\right]\right\}=\sum_i \varepsilon\theta_i+\frac{\varepsilon}{2}\frac{\partial\sum_i\theta_i}{\partial t} \tag{39}$$

In order to obtain eq. (13), meanwhile considering the fractional-order diffusion problem which does not involve macroscopic velocity, since equilibrium distribution function satisfies the conservation law of mass and momentum, we assume that the equilibrium distribution function $f_i^{\text{eq}}(x, t)$ satisfies the following conditions:

$$\sum_i f_i^{\text{eq}}=\sum_i f_i=u,\quad \sum_i c_i f_i^{\text{eq}}=0,\quad \sum_i c_i c_i f_i^{\text{eq}}=R\tilde{u},\quad \sum_i \varepsilon\theta_i+\frac{\varepsilon}{2}\frac{\partial\sum_i\theta_i}{\partial t}=f(u,x,t) \tag{40}$$

So eq. (40) can be put in the following form:

$$\frac{\partial u(x,t)}{\partial t}+\varepsilon\left(\frac{1}{2}-\tau\right)\frac{\partial^2 R\tilde{u}}{\partial x^2}=f(u,x,t) \tag{41}$$

Comparing eq. (15) with eq. (13), we can get the relationship between λ and τ:

$$\lambda=R\varepsilon\left(\frac{1}{2}-\tau\right) \tag{42}$$

Appealing to eqs. (15) and (42), we have:

$$\tau=\frac{1}{2}-\frac{k_\mu h^{2-\alpha}}{4R\varepsilon\Gamma(3-\alpha)\cos\dfrac{\alpha\pi}{2}} \tag{43}$$

The equilibrium distribution function $f_i^{\text{eq}}(x, t)$ can be obtained from eqs. (40):

$$f_1^{\text{eq}}=u-\frac{R\tilde{u}}{c^2},\qquad f_2^{\text{eq}}=f_3^{\text{eq}}=\frac{R\tilde{u}}{2c^2},\qquad \theta_{1,2,3}=\frac{f(u,x,t)}{3\varepsilon} \tag{44}$$

Numerical examples

In this section, the backward scheme is used for computing $\partial\theta_i/\partial t$ in eq. (25), and all the computations are computed by MATLAB 2013a software. *Example 1*, we present the approximate results of FLBM for fractional diffusion equation with an analytical solution. By comparing the numerical solution and analytical solution, the accuracy and reliability of the model are verified. In *Examples 2* and *3*, the FLBM is applied to solve RSFRDE with different non-linear source terms to illustrate the model application scope. Moreover, our numerical solutions of simulation consist with the reported results, which also illustrates the feasibility of our method.

The global relative error (GRE) and the maximum absolute error (MAE) are given for testing error precision of the model:

$$\mathrm{GRE}(t) = \frac{\sum_i \left| u(x_i,t) - u^*(x_i,t) \right|}{\sum_i \left| u^*(x_i,t) \right|} \tag{45}$$

$$\mathrm{MAE}(t) = \max_i \left| u(x_i,t) - u^*(x_i,t) \right| \tag{46}$$

where $u(x_i, t)$ and $u^*(x_i, t)$ are numerical and analytical solutions at (x_i, t), respectively.

Example 1. Consider the following space Riesz fractional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + f(x,t), \qquad x \in (0,1)$$

with the source term:

$$f(x,t) = (1+t)^{\alpha-1}(x-1)^2 x^2 \alpha + \frac{1}{\Gamma(5-\alpha)} \cdot$$

$$\cdot \left\{ \left(\frac{1+t}{1-x} \right)^\alpha (x-1)^2 \left[12x^2 - 6x\alpha + (\alpha-1)\alpha \right] + (1+t)^\alpha x^{2-\alpha} \left[12(x-1)^2 + (6x-7)\alpha + \alpha^2 \right] \right\} \sec\left(\frac{\pi\alpha}{2} \right)$$

and the initial data $u(x, 0) = x^2(1 - x)^2$, the boundary condition $u(0, t) = u(1, t) = 0$. It has the analytical solution $u(x, 0) = (t + 1)^\alpha x^2(1 - x)^2$.

In tab. 1, we obtained the GRE between the exact solution and the numerical solution for different α at $t = 0.01, 0.1, 0.5, 1$ with the mesh size $h = 0.01$. Table 2 also lists the MAE with $\alpha = 1.1, 1.3,\ldots, 1.99$ at $t = 0.01, 0.1, 0.5, 1$. From fig. 1(a) and 1(b), we can see that the numerical solution and analytical solution is consistent. Besides, from fig. 1(c) we can find that the FLBM is first-order accurate, this is because the order of convergence for the global spatial correlation process is $3 - \alpha$, and the boundary conditions is first-order accurate.

Example 2 Considering the RSFRDE with non-linear source term:

$$\frac{\partial u(x,t)}{\partial t} = \lambda \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + \sin u, \quad 0 < x < \pi$$

with the boundary condition $u(0, t) = u(\pi, t) = 0$, $0 \le t \le T$, and initial condition $u(x, 0) = \sin x$. In order to demonstrate the efficiency of our model, the results of the FLBM are compared with the ones of [26] in which fractional method of lines (FMOL) is used. The FMOL, as a classical method for solving fractional differential equations, was first introduced by Liu e*t al.* [27].

From fig. 2, it can be seen that the characters of reaction-diffusion system response with non-linear source term at the different time t with $\alpha = 1.8$, $N = 20$, $\lambda = 0.1$. Table 3 lists the numerical results of our FLBM with $\alpha = 1.8$, $\lambda = 0.1$, and $t = 1$. The last column of tab. 3 is the numerical results of FMOL with $\alpha = 1.8$, $\lambda = 0.1$, $t = 1$, and $h = \pi/200$ in [26]. From tab. 3, the numerical results of the two methods are very close.

Table 1. Global relative errors

t	α = 1.1	α = 1.3	α = 1.5	α = 1.8	α = 1.9	α = 1.99
0.01	0.0029	0.0018	0.0015	0.0005	0.0005	0.00021
0.1	0.0132	0.0060	0.0037	0.0021	0.0020	0.00019
0.5	0.0176	0.0069	0.0039	0.0035	0.0028	0.00015
1	0.0182	0.0070	0.0039	0.0036	0.0029	0.00016

Table 2. Maximum absolute errors

t	α = 1.1	α = 1.3	α = 1.5	α = 1.8	α = 1.9	α = 1.99
0.01	3.5418e-04	2.4083e-04	1.8400e-04	7.7488e-05	5.6021e-05	1.5563e-05
0.1	1.2000e-03	5.4134e-04	3.2183e-04	1.0961e-04	1.5722e-04	1.2563e-05
0.5	2.1030e-03	9.0018e-04	5.3067e-04	2.9491e-04	4.0555e-04	2.1748e-05
1	3.0130e-03	1.3012e-03	8.2268e-04	5.2616e-04	7.2377e-04	3.8906e-05

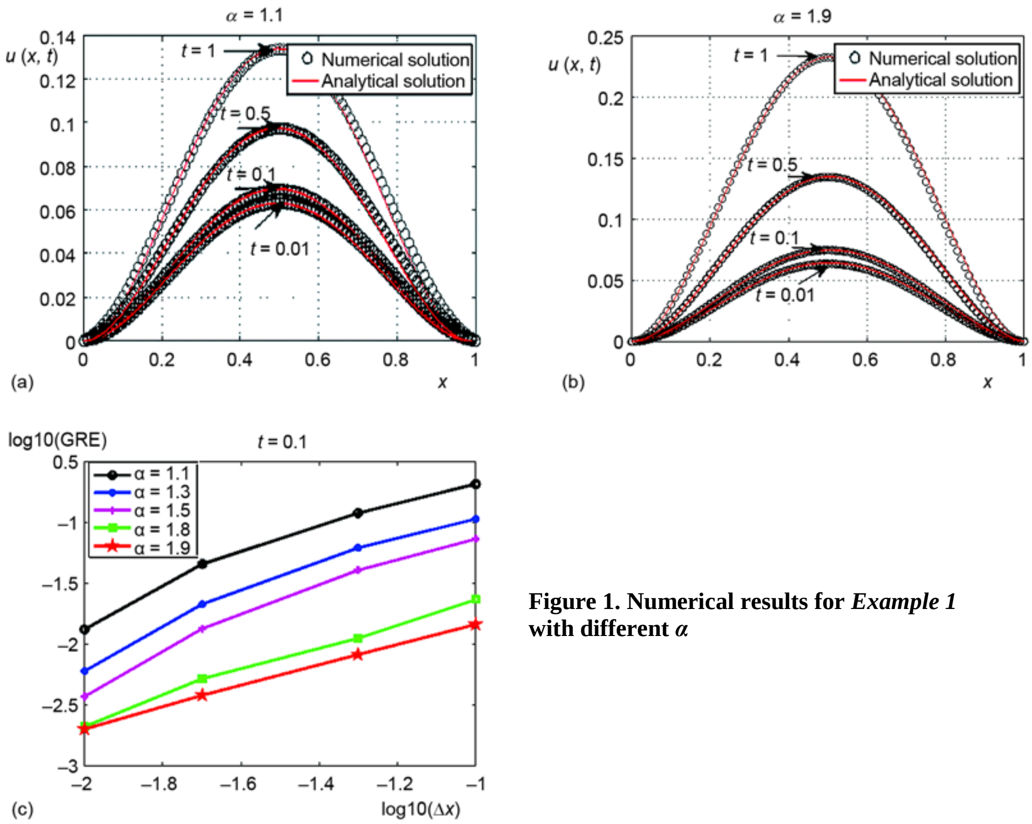

Figure 1. Numerical results for *Example 1* with different α

Example 3. Consider the following fractional order Fisher's equation [28], which is applied to model the growth and spread of species:

$$\frac{\partial u(x,t)}{\partial t} = \lambda \frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} + \mu u \left(1 - \frac{u}{k}\right),$$

$$1 < \alpha \leq 2, \qquad 0 < x < l, \qquad 0 \leq t \leq T$$

with boundary condition $u(0, t) = u(s, t) = 0$ and initial condition $u(x, 0) = v(x)$. The μ is intrinsic growth rate, k – the carrying capacity, and $v(x)$ – a step-like initial function which takes the constant value $v(50) = 0$ around the origin and rapidly decays to 0 away from the origin.

We take $\lambda = 0.1$, $k = 1$, $l = 100$, and choose the following equation as the initial function:

$$v(x) = 0.8\exp(-|x-50|), \qquad 0 \leq x \leq 100$$

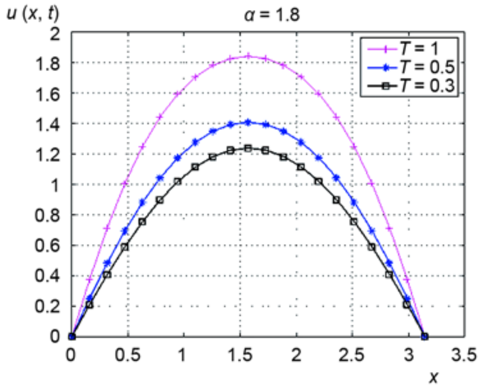

Figure 2. The state of the non-linear diffusion system with α = 1.8, N = 20, λ = 0.1 at the different t in *Example 2*

Table 3. Numerical results of FLBM and FMOL for different N with α = 1.8, λ = 0.1, and t = 1

X	FLBM			FMOL in [27]
	$h = \pi/20$	$h = \pi/100$	$h = \pi/200$	$h = \pi/200$
0.0000	0.00000	0.00000	0.00000	0.00000
0.3142	0.72989	0.72743	0.71297	0.71312
0.6283	1.25645	1.2541	1.24791	1.24780
0.9425	1.59814	1.59763	1.59367	1.59121
1.2566	1.78435	1.78402	1.78048	1.77773
1.5708	1.84300	1.84272	1.83558	1.83648
1.8850	1.78434	1.78402	1.78048	1.77783
2.1991	1.59814	1.59763	1.59367	1.59176
2.5133	1.25645	1.25541	1.25091	1.25101
2.8274	0.69853	0.72743	0.72707	0.72768
3.1416	0.00000	0.00000	0.00000	0.00000

Figure 3(a) shows the characteristics of the fractional order Fisher's equation at different time, T, with $\alpha = 1.8$, $\mu = 0.1$, and fig. 3(b) shows the features of the diffusion system response with different growth rate of invasive state μ at $t = 32$ with $\alpha = 1.8$. Figure 3(c) shows heaver tails and faster spreading with the fractional order α decreases. It is found that our simulation results of the fractional Fisher's equation well agree with the results of [28] excellently.

Conclusions

In this paper, a new FLBM for the RSFRDE with non-linear source term is proposed. Firstly, a numerical approximation for the global spatial correlation of Riesz fractional derivative is presented by using numerical discretization technique. Meanwhile, the local error

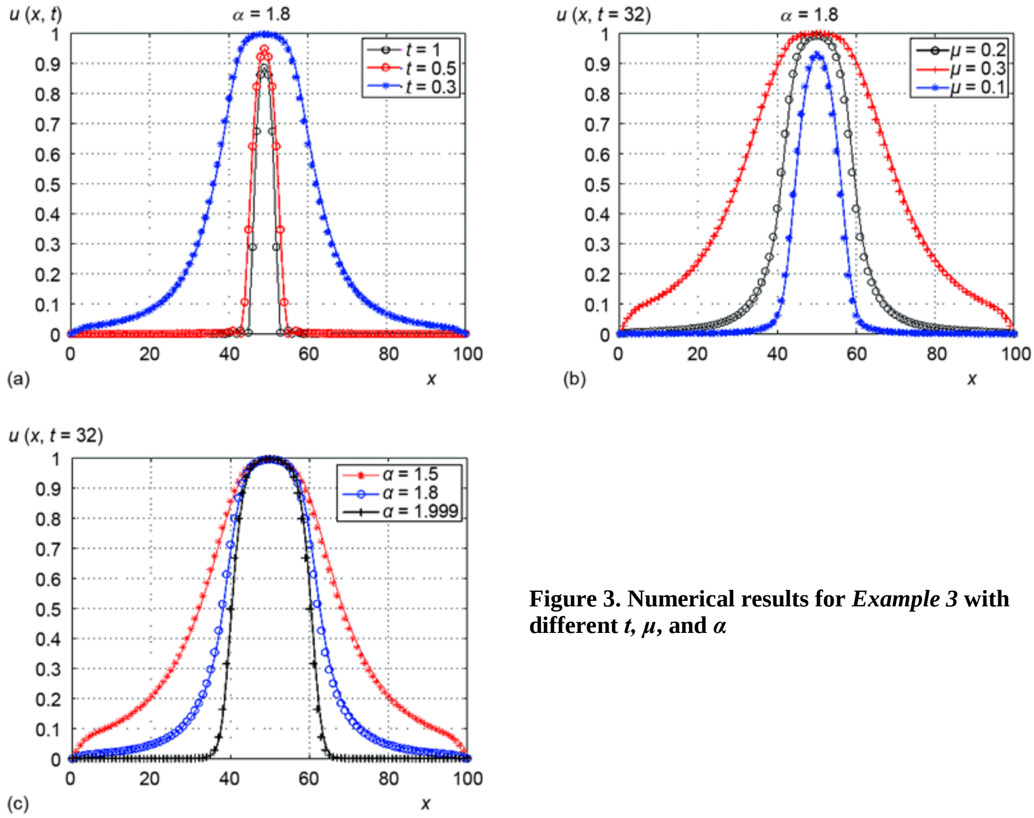

Figure 3. Numerical results for *Example 3* with different *t*, *μ*, and *α*

of the global spatial correlation process is estimated and proved. The FLBM is described and demonstrated by inserting global spatial correlation process into the evolution process of the standard LBM. Secondly, in order to recover the non-linear source term with the second order truncation error, we rewrite the evolution equation of FLBM by adding first-order and second-order terms. Finally, some numerical results of FLBM are presented. These numerical results are given to demonstrate that our FLBM is a computationally efficient method for RSFRDE with non-linear source term. Comparing with other previous methods, the proposed method is rather easy to practice.

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