SYMMETRY ANALYSIS OF A (2+1)-D SYSTEM

by

Yan WANG[a,b*] and Zhong-Zhou DONG[b]

[a] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan, China
[b] School of Science, Tianjin University of Commerce, Tianjin, China

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The classical Lie group method and the (2+1)-D generalized symmetry method in vector analysis are adopted to find infinitesimal symmetries for a (2+1)-D generalized Painleve Burgers system, and its various reduced systems are obtained.

Key words: *classical Lie group, (2+1)-D generalized symmetry, vector analysis*

Introduction

There are a number of fields such as thermal science, optics, physics, and chemistry in which non-linear differential equations are widely used to describe complex phenomena. Accordingly, the research of non-linear evolution equations has attracted much attention world-widely [1-8]. Lie group theory is an effective and direct approach to constructing exact solutions of non-linear evolution equations [9, 10]. It has been widely considering that symmetry plays a very important role in various scientific fields, especially in the soliton theory. So far there are many powerful methods to find the symmetry of PDE, such as the classical Lie group method [11-18], the generalized symmetry method [18], the non-classical Lie group method [13], and the Clarkson and Kruskal [12] direct method.

In fact, vector analysis is a completely new approach to symmetry [18]. The main target of this paper is to extend the vector analysis method to (2+1)-D case, and a novel modification for the generalized symmetry method is proposed.

Burgers equation [19] describes the far field of wave propagation in non-linear physics. This paper considers its extension, the (2+1)-D Painleve integrable Burgers equations:

$$\begin{cases} -u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0, \\ u_x - v_y = 0 \end{cases} \tag{1}$$

which were derived from the generalized Painleve integrability classification by Hong *et al.* [19], where α and β are constants.

This paper considers the following (2+1)-D generalized Painleve Burgers system:

$$\begin{cases} \Delta_1 \quad = u_t - \alpha\beta u_{xx} - \beta u_{yy} - G(x,y,t,u,v) = 0, \\ \Delta_2 \quad = v_t - \dfrac{1}{\gamma}(u_x - v_y) = 0 \end{cases} \tag{2}$$

where α, β, and γ are constants, with $\gamma \neq 0$.

* Corresponding author, e-mail: yanwang0125@163.com

The infinitesimal symmetries arising from the classical Lie group method and the (2+1)-D generalized symmetry method in vector analysis

The classical Lie group method

Suppose $(x, y, t) \in R^3$ are independent variables, $u, v \in R^3$ the dependent variable, and $u^{(l)}(x, y, t)$, $v^{(l)}(x, y, t)$ denote the set of all the partial derivatives of order l of u, v, respectively. Apply the classical Lie group method [13] to the following general N^{th} order PDE:

$$\begin{cases} \Delta_1 = \Delta_1\left[x,y,t,u,v,u^{(1)}(x,y,t),...,u^{(N)}(x,y,t),v^{(1)}(x,y,t),...,v^{(N)}(x,y,t)\right]=0, \\ \Delta_2 = \Delta_2\left[x,y,t,u,v,u^{(1)}(x,y,t),...,u^{(N)}(x,y,t),v^{(1)}(x,y,t),...,v^{(N)}(x,y,t)\right]=0 \end{cases} \tag{3}$$

We consider the one-parameter Lie group of infinitesimal transformations in (x, y, t, u, v) given by:

$$\begin{cases} x^* &= x+\in X+O(\in^2), \\ y^* &= y+\in Y+O(\in^2), \\ t^* &= t+\in T+O(\in^2), \\ u^* &= u+\in \varPhi_1+O(\in^2), \\ v^* &= v+\in \varPhi_2+O(\in^2) \end{cases} \tag{4}$$

where $\in$ is the group parameter, $X = X(x, y, t, u, v)$, $Y = Y(x, y, t, u, v)$, $T = T(x, y, t, u, v)$ $\varPhi_1 = \varPhi_1(x, y, t, u, v)$, and $\varPhi_2 = \varPhi_2(x, y, t, u, v)$. Requiring that eq. (3) is invariant under this transformation yields an overdetermined linear system of equations for the infinitesimals X, Y, T, $\varPhi_1$, and $\varPhi_2$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form:

$$v = X\partial_x + Y\partial_y + T\partial_t + \varPhi_1\partial_u + \varPhi_2\partial_v \tag{5}$$

where $\partial_x = \partial/\partial x$, *etc.*

To apply the classical Lie group method to eq. (2), we require that the set $S = \{(u, v)|\Delta_1 = 0, \Delta_2 = 0\}$ of solutions of eq. (2) is invariant under the transformation (4). This yields the determining PDE for X, Y, T, $\varPhi_1$, $\varPhi_2$ and is accomplished by requiring that:

$$\begin{cases} pr^{(2)}v(\Delta_1)\,|_{\Delta_1=0,\Delta_2=0} = 0, \\ pr^{(1)}v(\Delta_2)\,|_{\Delta_1=0,\Delta_2=0} = 0 \end{cases} \tag{6}$$

where $pr^{(1)}v$, and $pr^{(2)}v$ are given by:

$$\begin{cases} pr^{(2)}v &= v + \varPhi_1^{[t]}\partial_{u_t} + \varPhi_1^{[xx]}\partial_{u_{xx}} + \varPhi_1^{[yy]}\partial_{u_{yy}} \\ pr^{(1)}v &= v + \varPhi_2^{[t]}\partial_{v_t} + \varPhi_1^{[x]}\partial_{u_x} + \varPhi_2^{[y]}\partial_{v_y} \end{cases} \tag{7}$$

where

$$\Phi_1^{[x]} = \mathrm{D}_x(\Phi_1 - Xu_x - Yu_y - Tu_t) + Xu_{xx} + Yu_{xy} + Tu_{xt}$$

$$\Phi_2^{[y]} = \mathrm{D}_y(\Phi_2 - Xv_x - Yv_y - Tv_t) + Xv_{xy} + Yv_{yy} + Tv_{yt}$$

$$\Phi_1^{[t]} = \mathrm{D}_t(\Phi_1 - Xu_x - Yu_y - Tu_t) + Xu_{xt} + Yu_{yt} + Tu_{tt}$$

$$\Phi_2^{[t]} = \mathrm{D}_t(\Phi_2 - Xv_x - Yv_y - Tv_t) + Xv_{xt} + Yv_{yt} + Tv_{tt}$$

$$\Phi_1^{[xx]} = \mathrm{D}_{xx}(\Phi_1 - Xu_x - Yu_y - Tu_t) + Xu_{xxx} + Yu_{xxy} + Tu_{xxt}$$

$$\Phi_1^{[yy]} = \mathrm{D}_{yy}(\Phi_1 - Xu_x - Yu_y - Tu_t) + Xu_{xyy} + Yu_{yyy} + Tu_{yyt}$$

and D_x, *etc.* are total derivative operators.

Hence we obtain the following determining equations for the infinitesimals and have the following general solution:

$$X = \frac{c_1 x}{2} + c_5$$

$$Y = \frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4$$

$$T = c_1 t + c_2$$

$$\Phi_1 = \left(\frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} - c_3\right)u + r_2(t,x,y)$$

$$\Phi_2 = \left(-\frac{c_1}{2} + \frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} - c_3\right)v + r_1(t,x,y)$$

where $c_i = (i = 1, 2, \ldots, 5)$ are arbitrary constants, $r_1 = r_1(t, x, y)$, $r_2 = r_2(t, x, y)$ are arbitrary functions, and they satisfy the following relation:

$$-c_1 u - 4r_{2t}\gamma^2\beta + 4G_t\gamma^2\beta c_1 t + 4G_t\gamma^2\beta c_2 + 2G_x\gamma^2\beta c_1 x + 4G_x\gamma^2\beta c_5 + 2G_y\gamma^2\beta c_1 y +$$
$$+ 2G_y\gamma\beta c_1 t + 4G_y\gamma^2\beta c_4 - G_u u c_1 y\gamma + G_u u c_1 t + 4G_u u c_3\gamma^2\beta + 4G_u r_2\gamma^2\beta +$$
$$+ 2G_v v c_1\gamma^2\beta - G_v v c_1 y\gamma + G_v v c_1 t + 4G_v v c_3\gamma^2\beta + 4G_v r_1\gamma^2\beta + Gc_1 y\gamma -$$
$$-Gc_1 t - 4Gc_3\gamma^2\beta + 4c_1\gamma^2 G\beta + 4\beta^2\gamma^2 r_{2yy} + 4\alpha\beta^2\gamma^2 r_{2xx} = 0$$

$$r_{1t}\gamma - r_{2x} + r_{1y} = 0$$

The associated vector field is:

$$\mathrm{v} = \left(\frac{c_1 x}{2} + c_5\right)\partial_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)\partial_y + (c_1 t + c_2)\partial_t +$$

$$+\left[\left(\frac{c_1 y}{4\beta\gamma}-\frac{c_1 t}{4\beta\gamma^2}-c_3\right)u+r_2\right]\partial_u+\left[\left(-\frac{c_1}{2}+\frac{c_1 y}{4\beta\gamma}-\frac{c_1 t}{4\beta\gamma^2}-c_3\right)v+r_1\right]\partial_v \tag{8}$$

The (2+1)-D generalized symmetry method in vector analysis

In this section, we investigate the theory of generalized symmetry [18]. Here we introduce some notations. Assume $(x, y, t) \in R^3$ is independent variable and $u = (u^1, u^2) \in R^2$ is dependent variable. We denote the set of smooth differential functions $P(x, y, t, u^{(n)})$ by Ã, depending on x, y, t, u and derivatives of u up to some finite, but unspecified order n, where the range of P is R. Suppose $P[u] := P(x, y, t, u^{(n)})$.

Definition 1. A generalized vector field is:

$$\mathrm{V}=\xi[u]\frac{\partial}{\partial x}+\eta[u]\frac{\partial}{\partial y}+\tau[u]\frac{\partial}{\partial t}+\Psi_1[u]\frac{\partial}{\partial u^1}+\Psi_2[u]\frac{\partial}{\partial u^2} \tag{9}$$

where $\xi[u], \eta[u], \tau[u], \Psi_1[u]$, and $\Psi_2[u] \in \tilde{\mathrm{A}}$.

Definition 2. If V is given as in eq. (9), we set:

$$Q_i[u]=\Psi_i[u]-\xi[u]u_x^i-\eta[u]u_y^i-\tau[u]u_t^i \tag{10}$$

where $u_x^i = \partial u^i/\partial x, u_y^i = \partial u^i/\partial y, u_t^i = \partial u^i/\partial t\ (i=1, 2)$, and the 2-tuple $Q[u]=(Q_1[u], Q_2[u])$ is referred to as the characteristic of the vector field, eq. (9).

Definition 3. Consider $Q[u]=(Q_1[u], Q_2[u]) \in \tilde{\mathrm{A}}^2$. The generalized vector field:

$$\mathrm{V}_Q=Q_1[u]\frac{\partial}{\partial u^1}+Q_2[u]\frac{\partial}{\partial u^2} \tag{11}$$

is called an evolutionary vector field. If $Q[u] = (Q_1[u], Q_2[u])$ takes the form of eq. (10) then the generalized vector field V_Q is called an evolutionary representative of V, and Q is called V_Q's characteristic.

Lemma 1. A generalized vector field V is a symmetry of a system of differential equations if and only if its evolutionary representative V_Q is a symmetry.

Now suppose a system of n^{th} order evolution equations:

$$\left(\frac{\partial u^1}{\partial t},\frac{\partial u^2}{\partial t}\right)=\frac{\partial u}{\partial t}=P[u] \tag{12}$$

where $P[u]=(P_1[u],P_2[u]) \in \tilde{\mathrm{A}}^2$ depends on $x, y \in R$, $u \in R^2$ with x-derivatives and y-derivatives of u up to some finite order n. In order to find generalized symmetries V of the previous evolution equation, we can replace V by its evolutionary representative V_Q, where Q satisfies eq. (10), by *Lemma 1*.

An evolutionary vector field V_Q is a symmetry of the system of evolution eq. (12) if and only if:

$$\mathrm{D}_t Q_i = pr\mathrm{V}_Q(P_i), \qquad i=1,2 \tag{13}$$

where D_t denotes the total derivative operator and is defined by:

$$\mathrm{D}_t = \frac{\partial}{\partial t} + u_t^1 \frac{\partial}{\partial u^1} + u_t^2 \frac{\partial}{\partial u^2} + u_{xt}^1 \frac{\partial}{\partial u_x^1} + u_{xt}^2 \frac{\partial}{\partial u_x^2} + u_{yt}^1 \frac{\partial}{\partial u_y^1} + u_{yt}^2 \frac{\partial}{\partial u_y^2} + u_{tt}^1 \frac{\partial}{\partial u_t^1} + u_{tt}^2 \frac{\partial}{\partial u_t^2} + \ldots \tag{14}$$

$pr\mathrm{V}_Q$ is the prolongation of the evolutionary vector field V_Q given by:

$$pr\mathrm{V}_Q(P_i) = P'_{iu^1} Q_1 + P'_{iu^2} Q_2 \tag{15}$$

where $P_i \in \tilde{\mathrm{A}}$, and $P'_{iu^i} Q_i$ means:

$$P'_{iu^i} Q_i = \frac{\partial P_i}{\partial u^i} Q_i + \frac{\partial P_i}{\partial u_x^i} \mathrm{D}_x Q_i + \frac{\partial P_i}{\partial u_{xx}^i} \mathrm{D}_{xx} Q_i + \ldots, \qquad i = 1, 2 \tag{16}$$

Consequently, one can show that the eq. (13) can be written:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}_t = \begin{pmatrix} P'_{1u^1} & P'_{1u^2} \\ P'_{2u^1} & P'_{2u^2} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \tag{17}$$

Lemma 2. For evolutionary vector field V_Q being a symmetry of the system of evolution eq. (12), then u is the group invariant solution of eq. (12) corresponding V_Q if and only if u satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial t} = P[u], \\ Q[u] = 0 \end{cases} \tag{18}$$

where $Q[u]$ is the characteristic of the vector field V_Q.

Next we apply the (2+1)-D generalized symmetry method in vector analysis to eq. (2). From eqs. (2) and (12) it is easily shown that:

$$P_1 = \alpha\beta u_{xx} + \beta u_{yy} + G(t, x, y, u, v), \qquad P_2 = \frac{u_x - v_y}{\gamma}$$

$$P'_{1u} = \alpha\beta \mathrm{D}_{xx} + \beta \mathrm{D}_{yy} + G_u, \qquad P'_{1v} = G_v$$

$$P'_{2u} = \frac{\mathrm{D}_x}{\gamma}, \qquad P'_{2v} = -\frac{\mathrm{D}_y}{\gamma}$$

where D_x, D_y, D_{xx}, and D_{yy} are total derivative operators, $G_u = \partial G/\partial u$ and $G_v = \partial G/\partial v$.

In order to make the calculation easy, we seek the symmetry of system eq. (2) in the form:

$$\begin{cases} Q_1 = a_1 u_x + a_2 v_x + b_1 u_t + b_2 v_t + c_1 u_y + c_2 v_y + d_1 u + d_2 v + m \\ Q_2 = e_1 u_x + e_2 v_x + f_1 u_t + f_2 v_t + g_1 u_y + g_2 v_y + h_1 u + h_2 v + n \end{cases} \tag{19}$$

where a_i, b_i, c_i, d_i, e_i, f_i, g_i, and h_i, (i = 1, 2) are undetermined functions of x, y, t, u, and v, and m and n are undetermined functions of x, y, and t. Equation (17) can be reformed:

$$\begin{cases} Q_{1t} = P'_{1u}Q_1 + P'_{1v}Q_2, \\ Q_{2t} = P'_{2u}Q_1 + P'_{2v}Q_2, \end{cases} \tag{20}$$

by system of eq. (2). Put eq. (2) into eq. (20), and replace u_t by $\alpha\beta u_{xx} + \beta u_{yy} + G(t,x,y,u,v)$, v_t by $(u_x - v_y)/\gamma$, respectively. Therefore, by equating the coefficients of the various monomials in fourth, third, second, and first order partial derivatives of u and v, we obtain new systems of PDE which are calculated with a mathematical software and we get the following solutions:

$$a_1 = a_1(t,x,y,u,v), \qquad a_2 = 0$$

$$b_1 = c_1 t + c_2, \qquad b_2 = \left(-a_1 + \frac{c_1 x}{2} + c_5\right)\gamma$$

$$c_1 = \frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4, \qquad c_2 = \frac{c_1 x}{2} + c_5 - a_1$$

$$d_1 = d_1(t,x,y,u,v), \qquad d_2 = \frac{-ud_1 + \left(\dfrac{c_1 y}{4\beta\gamma} - \dfrac{c_1 t}{4\beta\gamma^2} + c_3\right)u + f}{v}$$

$$e_1 = e_1(t,x,y,u,v), \qquad e_2 = \frac{c_1 x}{2} + c_5$$

$$f_1 = 0, \qquad f_2 = -e_1\gamma + c_1 t + c_2$$

$$g_1 = 0, \qquad g_2 = -e_1 + \frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4$$

$$h_1 = \frac{-vh_2 - \dfrac{c_1 v}{2} + \dfrac{c_3 yv}{4\beta\gamma} + v\left(-\dfrac{c_1 t}{4\beta\gamma^2} + c_3\right) + g}{u}, \qquad h_2 = h_2(t,x,y,u,v)$$

$$m = m(t,x,y), \qquad n = n(t,x,y)$$

where c_i (i = 1, 2,…, 5) are arbitrary constants, $f = f(t, x, y)$ and $g = g(t, x, y)$ are arbitrary functions, and they satisfy the following relation:

$$(g+n)_t\gamma - (f+m)_x + (g+n)_y = 0$$

$$\begin{aligned} &-c_1 u - 4G_u\gamma^2 m\beta - 4G_v\gamma^2 n\beta + 4c_1\gamma^2 G\beta + 4\gamma^2 G_x\beta c_5 + 4\gamma^2 G_t\beta c_2 + 4\gamma^2 G_y\beta c_4 + Gc_1 y\gamma + \\ &+ 4Gc_3\gamma^2\beta + G_u u c_1 t - 4G_u f\gamma^2\beta + G_v v c_1 t - 4G_v g\gamma^2\beta - 4\alpha\beta^2\gamma^2 m_{xx} - 4\alpha\beta^2\gamma^2 f_{xx} + \\ &+ 2\gamma^2 G_x\beta c_1 x + 4\gamma^2 G_t\beta c_1 t + 2\gamma^2 G_y\beta c_1 y + 2\gamma G_y\beta c_1 t - G_u u c_1 y\gamma - 4G_u u c_3\gamma^2\beta + 2G_v v c_1\gamma^2\beta - \\ &- G_v v c_1 y\gamma - 4G_v v c_3\gamma^2\beta + 4f_t\gamma^2\beta + 4m_t\gamma^2\beta - Gc_1 t - 4\beta^2\gamma^2 f_{yy} - 4\beta^2\gamma^2 m_{yy} = 0 \end{aligned}$$

Set $a_1 = e_2$, $f_2 = b_1$, and $g_2 = c_1$, from eq. (19) we obtain:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \left(\frac{c_1 x}{2} + c_5\right)u_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)u_y + \left(c_1 t + c_2\right)u_t \\ \left(\frac{c_1 x}{2} + c_5\right)v_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)v_y + \left(c_1 t + c_2\right)v_t \end{pmatrix} + \\ + \begin{pmatrix} \left(\frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)u + r_2 \\ \left(\frac{-c_1}{2} + \frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)v + r_1 \end{pmatrix} \tag{21}$$

where $r_2 = f + m$ and $r_1 = g + n$, are arbitrary functions and:

$$\mathrm{V} = \left(\frac{c_1 x}{2} + c_5\right)\frac{\partial}{\partial x} + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)\frac{\partial}{\partial y} + \left(c_1 t + c_2\right)\frac{\partial}{\partial t} + \left[\left(\frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)u + r_2\right]\frac{\partial}{\partial u} + \\ + \left[\left(\frac{-c_1}{2} + \frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)v + r_1\right]\frac{\partial}{\partial v} \tag{22}$$

Remark

As we can see eq. (8) obtained by the classical Lie group method, is same with eq. (22) by the (2-1)-D generalized symmetry method.

Applications of symmetries to similar reductions and exact solutions

In the rest of this paper, some symmetries are applied to reduce eq. (2). To make the calculation easy, we set $r_2 = 0$ and $r_1 = 0$. In this case from eq. (21) we get:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} \left(\frac{c_1 x}{2} + c_5\right)u_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)u_y + (c_1 t + c_2)u_t + \left(\frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)u \\ \left(\frac{c_1 x}{2} + c_5\right)v_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)v_y + (c_1 t + c_2)v_t + \left(\frac{-c_1}{2} + \frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)v \end{pmatrix} \tag{23}$$

Now set:

$$\left(\frac{c_1 x}{2} + c_5\right)u_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)u_y + (c_1 t + c_2)u_t + \left(\frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)u = 0 \tag{24}$$

$$\left(\frac{c_1 x}{2} + c_5\right)v_x + \left(\frac{c_1 y}{2} + \frac{c_1 t}{2\gamma} + c_4\right)v_y + (c_1 t + c_2)v_t + \left(\frac{-c_1}{2} + \frac{c_1 y}{4\beta\gamma} - \frac{c_1 t}{4\beta\gamma^2} + c_3\right)v = 0 \tag{25}$$

There are ten cases to be considered, which are listed as follows.

(1) When $c_1 = c_2 = c_3 = c_4 = c_5 = 1$,

$$\mathrm{V} = \left(\frac{x}{2}+1\right)\frac{\partial}{\partial x} + \left(\frac{y}{2}+\frac{t}{2\gamma}+1\right)\frac{\partial}{\partial y} + (t+1)\frac{\partial}{\partial t} + \left[\left(\frac{y}{4\beta\gamma} - \frac{t}{4\beta\gamma^2}+1\right)u\right]\frac{\partial}{\partial u} + \\ + \left[\left(\frac{1}{2}+\frac{y}{4\beta\gamma}-\frac{t}{4\beta\gamma^2}\right)v\right]\frac{\partial}{\partial v}$$

$$G = \frac{ut}{4\beta\gamma^2(t+1)} + (t+1)^{\left(-2-\frac{1}{2\beta\gamma^2}+\frac{1}{2\beta\gamma}\right)}$$

$$G_1\left[\frac{x+2}{\sqrt{t+1}}, \frac{y\gamma-2+2\gamma-t}{\sqrt{t+1}\gamma}, \quad u(t+1)^{\frac{1-\gamma+2\beta\gamma^2}{2\gamma^2\beta}}\mathrm{e}^{\frac{y\gamma-2+2\gamma-t}{2\gamma^2\beta}}, \quad v(t+1)^{\frac{-\beta\gamma^2-1+\gamma}{2\gamma^2\beta}}\mathrm{e}^{\frac{y\gamma-2+2\gamma-t}{2\gamma^2\beta}}\right]\mathrm{e}^{\frac{y\gamma-2+2\gamma-t}{2\beta\gamma^2}}$$

By eqs. (24) and (25), we can obtain:

$$u = (t+1)^{-\frac{1-\gamma+2\gamma^2\beta}{2\gamma^2\beta}}\mathrm{U}\left(\frac{x+2}{\sqrt{t+1}}, \quad \frac{y\gamma-2+2\gamma-t}{\sqrt{t+1}\gamma}\right)\mathrm{e}^{-\frac{y\gamma-2+2\gamma-t}{2\gamma^2\beta}}$$

$$v = (t+1)^{-\frac{1}{2}-\frac{1}{2\gamma^2\beta}+\frac{1}{2\gamma\beta}}\mathrm{V}\left(\frac{x+2}{\sqrt{t+1}}, \quad \frac{y\gamma-2+2\gamma-t}{\sqrt{t+1}\gamma}\right)\mathrm{e}^{-\frac{y\gamma-2+2\gamma-t}{2\gamma^2\beta}}$$

where

$$x = -2+\xi\sqrt{t+1}, \qquad y = \frac{\eta\sqrt{t+1}\gamma+2-2\gamma+t}{\gamma}$$

Replace G, u, v and x, y into eq. (2), then we reduce the (2+1)-D system to (1+1)-D one:

$$\begin{cases} 4\beta\gamma^2G_1 + 4U_{\eta\eta}\beta^2\gamma^2 + 2\beta\gamma^2U_\xi\xi + 2\beta\gamma^2U_\eta\eta + U - 2U\gamma + 4U\beta\gamma^2 + 4\alpha\beta^2U_{\xi\xi}\gamma^2 = 0 \\ -V\gamma + 2V\beta\gamma^2 + V - V\beta\gamma^2 + 2\beta\gamma U_\xi + \beta\gamma^2\xi V_\xi + \beta\gamma^2\eta V_\eta = 0 \end{cases} \tag{26}$$

where $G_1 = G_1(\xi, \eta, U, V)$, $U = U(\xi, \eta)$, $V = V(\xi, \eta)$.

(2) When $c_1 = 0$, $c_2 = c_3 = c_4 = c_5 = 1$:

$$\mathrm{V} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}$$

$$G = G_1(-t+x, -t+y, u\mathrm{e}^t, v\mathrm{e}^t)\mathrm{e}^{-t}$$

$$u = U(-t + x, -t + y)e^{-t}$$

$$v = V(-t + x, -t + y)e^{-t}$$

where $x = t + \xi$, $y = t + \eta$. Replace G, u, v and x, y into eq. (2), and we get:

$$\begin{cases} U_\xi + U_\eta + U + \alpha\beta U_{\xi\xi} + \beta U_{\eta\eta} + G_1 = 0 \\ -\gamma V_\xi - \gamma V_\eta - \gamma V - U_\xi + V_\eta = 0 \end{cases} \tag{27}$$

(3) When $c_1 = c_2 = 0$, $c_3 = c_4 = c_5 = 1$:

$$V = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad G = G_1(t, -x + y, ue^x, ve^x)e^{-x},$$
$$u = U(t, -x + y)e^{-x}, \quad v = V(t, -x + y)e^{-x}$$

where $t = \xi$, $y = x + \eta$. Replace G, u, v and t, y into eq. (2), we can get:

$$\begin{cases} -U_\xi + \alpha\beta U_{\eta\eta} + 2\alpha\beta U_\eta + \alpha\beta U + \beta U_{\eta\eta} + G_1 = 0 \\ \gamma V_\xi + U_\eta + U + V_\eta = 0 \end{cases} \tag{28}$$

(4) When $c_1 = c_2 = c_5 = 0$, $c_3 = c_4 = 1$:

$$V = \frac{\partial}{\partial y} + u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad G = G_1(t, x, ue^y, ve^y)e^{-y}, \quad u = U(t, x)e^{-y}, \quad v = V(t, x)e^{-y}$$

where $t = \xi$, $x = \eta$. Put G, u, v and t, x into eq. (2), we obtain:

$$\begin{cases} -U_\xi + \alpha\beta U_{\eta\eta} + \beta U + G_1 = 0 \\ \gamma V_\xi - U_\eta - V = 0 \end{cases} \tag{29}$$

(5) When $c_1 = c_2 = c_3 = c_4 = 0$, $c_5 = 1$:

$$V = \frac{\partial}{\partial x}, \quad G = G_1(t, y, u, v), \quad u = u(t, y), \quad v = v(t, y), \quad t = \xi, \qquad y = \eta$$

Put these solutions into system eq. (2), we can reduce it to the following formula:

$$\begin{cases} u_t = \beta u_{yy} + G_1(t, y, u, v) = 0 \\ \gamma v_t = -v_y \end{cases} \tag{30}$$

Example 1. If $G = G_1(t, y, k, k) = 0$ for some constant k, then $u = k$, $v = k$ is x-translation invariant solution of eq. (2), where reaction term satisfies $G = G_1(t, y, k, k)$.

Example 2. If:

$$G = G_1(t, y, u, v) = A_1(t, y)u + A_2(t, y)v + A_3(t, y) \tag{31}$$

where $A_i(t, y)(i = 1, 2, 3)$ are continuous functions. Assume $[u_0(t, y), v_0(t, y)]$ is a solution of eq. (30), then $u = u_0(t, y)$, $v = v_0(t, y)$ is a x-translation invariant solution of eq. (2), where reaction term satisfies eq. (31).

(6) When $c_1 = c_2 = c_3 = c_5 = 0$, $c_4 = 1$:

$$V = \frac{\partial}{\partial y}, \quad G = G_1(t,x,u,v), \quad u = u(t,x), \quad v = v(t,x), \quad t = \xi, \qquad x = \eta$$

By these solutions, the system eq. (2) can be expressed:

$$\begin{cases} u_t = \alpha\beta u_{xx} + G_1(t,x,u,v) = 0 \\ \gamma v_t = u_x \end{cases} \tag{32}$$

Example 3. If $G = G_1(t, x, k, k) = 0$ for some constant k, then $u = k$, $v = k$ is y-translation invariant solution of eq. (2), where reaction term satisfies $G = G_1(t, x, k, k)$.

Example 4. If:

$$G = G_1(t,x,u,v) = B_1(t,x)u + B_2(t,x)v + B_3(t,x) \tag{33}$$

where $B_i(t, x)(i = 1, 2, 3)$ are continuous functions. Assume $[u_1(t, x), v_1(t, x)]$ is a solution of eq. (32), then $u = u_1(t, x)$, $v = v_1(t, x)$ is a y-translation invariant solution of eq. (2), where reaction term satisfies eq. (33).

(7) When $c_1 = c_3 = c_4 = c_5 = 0$, $c_2 = 1$:

$$V = \frac{\partial}{\partial t}, \quad G = G_1(x,y,u,v), \quad u = u(x,y), \quad v = v(x,y), \quad x = \xi, \qquad y = \eta$$

Replace these solutions into the formula eq. (2), we can achieve:

$$\begin{cases} \alpha\beta u_{xx} + \beta u_{yy} + G_1(x,y,u,v) = 0 \\ u_x - v_y = 0 \end{cases} \tag{34}$$

Example 5. If $G = G_1(x, y, k, k) = 0$ for some a constant k, then $u = k$, $v = k$ is a t-translation invariant solution of eq. (2), where reaction term satisfies $G = G_1(x, y, k, k)$.

Example 6. If:

$$G = G_1(x,y,u,v) = E_1(x,y)u + E_2(x,y)v + E_3(x,y) \tag{35}$$

where $E_i(x, y)(i = 1, 2, 3)$ are continuous functions. Assume $[u_2(x, y), v_2(x, y)]$ is a solution of eq. (34), then $u = u_2(x, y)$, $v = v_2(x, y)$ is a t-translation invariant solution of eq. (2), where reaction term satisfies eq. (35).

(8) When $c_2 = c_5 = 1$, $c_1 = c_3 = c_4 = 0$:

$$V = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}, \quad G = G_1(\xi,y,u,v), \quad u = p(\xi,y), \quad v = q(\xi,y), \quad \xi = x - t$$

Put this solution into system of eq. (2), we obtain the following equation:

$$\begin{cases} p_\xi + \alpha\beta p_{\xi\xi} + \beta p_{yy} + G_1(\xi, y, p, q) = 0 \\ \gamma q_\xi + p_\xi - p_y = 0 \end{cases} \tag{36}$$

and $u = p(x - t, y)$, $v = q(x - t, y)$ is a solution of eq. (2).

(9) When $c_2 = c_4 = 0$, $c_1 = c_3 = c_5 = 0$:

$$V = \frac{\partial}{\partial y} + \frac{\partial}{\partial t}, \quad G = G_1(x,\eta,u,v), \quad u = p_1(x,\eta), \quad v = q_1(x,\eta), \quad \eta = -t + y$$

By this solution, the system of eq. (2) can be represented:

$$\begin{cases} p_{1\eta} + \alpha\beta p_{1xx} + \beta p_{1\eta\eta} + G_1(x,\eta, p_1, q_1) = 0 \\ \gamma q_{1\eta} + p_{1x} - q_{1\eta} = 0 \end{cases} \tag{37}$$

and $u = p_1(x, -t + y)$, $v = q_1(x, -t + y)$ is a solution of eq. (2).

(10) When $c_4 = c_5 = 0$, $c_1 = c_2 = c_3 = 0$:

$$V = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad G = G_1(t,\eta,u,v), \quad u = p_2(t,\eta), \quad v = q_2(t,\eta), \quad \eta = y - x$$

By this solution the system eq. (2) can be represented:

$$\begin{cases} p_{2t} = \alpha\beta p_{2\eta\eta} + \beta p_{2\eta\eta} + G_1(t,\eta, p_2, q_2) = 0 \\ \gamma q_{2t} = -p_{2\eta} - q_{2\eta} \end{cases} \tag{38}$$

and $u = p_2(t, y - x)$, $v = q_2(t, y - x)$ is a solution of eq. (2).

Conclusion

In this paper we have reviewed and compared the classical Lie group method and the (2+1)-D generalized symmetry method in vector analysis as techniques for determining symmetry reductions of non-linear PDE, using the (2+1)-D generalized Painleve Burgers equations as the example. We find that these two methods coincide with each other. The (2+1)-D generalized symmetry method is discussed from the point of view of vector and in the present work the vector analysis method is generalized to (2+1)-D case for the first time. We reduce the (2+1)-D equations to (1+1)-D systems. A major difficulty in the determination of symmetry reductions is the large quantity of routine calculations involved. Although the symbolic manipulation programs have been developed, sometimes one has to solve very complex equations. In conclusion, the symmetry analysis based on Lie group is a very powerful method and is worthy of studying further.

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Nomenclature

x – space co-ordinate, [m]
t – time co-ordinate, [s]
y – space co-ordinate, [m]

Greek symbols

α, β, γ – constants, [–]
$u(x, t)$ – velocity, [ms^{-1}]
$v(x, t)$ – velocity, [ms^{-1}]

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