DIVERSITY SOLITON EXCITATIONS FOR THE (2+1)-DIMENSIONAL SCHWARZIAN KORTEWEG-DE VRIES EQUATION

by

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Original scientific paper
https://doi.org/10.2298/TSCI1804781L

With the aid of symbolic computation, we derive new types of variable separation solutions for the (2+1)-dimensional Schwarzian Korteweg-de Vries equation based on an improved mapping approach. Rich coherent structures like the soliton-type, rouge wave-type, and cross-like fractal type structures are presented, and moreover, the fusion interactions of localized structures are graphically investigated. Some of these solutions exhibit a rich dynamic, with a wide variety of qualitative behavior and structures that are exponentially localized.

Key words: *Schwarzian Korteweg-de Vries equations, rouge wave, improved mapping method, variable separation approach, cross-like fractal structures*

Introduction

In the line with the development of symbolic computation, much work has been focused on the various extensions and application of the known algebraic methods to construct solutions of non-linear evolution equations [1-4]. Special solutions, such as dromions, soliton excitations, and other coherent structures, which might be useful in explaining some phenomena in both mathematics and physics, were discussed in open literature [5-8]. The abundant localized coherent structures such as dromion, peakon, compacton, and ring soliton were investigated [9] and recently, there is much interest in integrable (2+1)-D equations, *i. e.*, equations with two spatial variables and one temporal variable [10-13].

In this paper, we pay our attention to construct a localized structure by a projective equation method [14] to obtain a variable separation solution of an integrable (2+1)-D equation, *i. e.*,

$$H_t+\frac{1}{4}H_{xxz}-\frac{H_xH_{xz}}{2H}-\frac{H_{xx}H_z}{4H}+\frac{H_x^2H_z}{2H^2}-\frac{1}{8}H_x\partial_x^{-1}\left(\frac{H_x^2}{H^2}\right)_z=0 \tag{1}$$

where $\partial_x^{-1}f=\int f\,\mathrm{d}x$.

Equation (1) was firstly introduced by Kudryashov and Pickering [15] for a flow problem, it is an extension of the well-known Korteweg-de Vries (KdV) equation. Equation (1) is generally called as Schwarzian Korteweg-de Vries equation or the Schwarz-Korteweg--de Vries equation.

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It is easy to see that eq. (1) is readily expressed as [16]:

$$\frac{\varphi_t}{\varphi_x} + S_{2+1}[\varphi; x] = 0 \tag{2}$$

where φ denotes relation $H = \varphi_x$ and $S_{2+1}[\varphi;x]$ is the Schwarz derivative of φ defined as:

$$S_{2+1}[\varphi; x] \equiv \left(\frac{\varphi_{xx}}{\varphi_x}\right)_z - \frac{1}{2}\partial_x^{-1}\left(\frac{\varphi_{xx}}{\varphi_x}\right)_z^2 \tag{3}$$

The integrability of eq. (1) is proved by Toda and Yu [17] in the sense of Weiss-Tabor-Carnevale Painleve expansion, and the solution properties has been investigated [18-21].

Variable separation solution for (2+1)-D SKdV Equation

In this section, the projective equation method is applied to study eq. (1). Although this (2+1)-D SKdV equation appears in a nonlocal form, by using the following transformations:

$$H = \varphi_x, \qquad \varphi = \exp(\Phi), \qquad \Phi_x = U, \qquad \Phi_t = V \tag{4}$$

we can rewrite eq. (3) in the following system of differential equations:

$$4U^2V_x - 4UU_xV + U^2U_{xxy} - UU_{xx}U_y - 3UU_xU_{xy} + 3U_x^2U_y - U^4U_y = 0 \tag{5}$$

$$U_t - V_x = 0 \tag{6}$$

Along with the projective equation method, we assume that eqs. (5) and (6) posses solutions in the form:

$$U = a_0(X) + \sum_{i=1}^{m} a_i(X)F^i[\psi(X)], \qquad V = b_0(X) + \sum_{j=1}^{n} b_j(X)F^j[\psi(X)] \tag{7}$$

where $a_0(X)$, $a_i(X)$, $b_0(X)$, $b_i(X)$ (i = 1, 2, …, m; j = 1, 2, …, n), and $\Psi(X)$ are all functions of X to be determined and $F[\Psi(X)]$ satisfies the Riccati equation:

$$F'[\psi(X)] = \delta + F^2[\psi(X)] \tag{8}$$

with the following solutions:

(1) When $\delta = 0,\ F[\Psi(X)] = -1/[\Psi(X)]$
(2) When $\delta < 0,\ F[\Psi(X)] = -\sqrt{-\delta}\tanh\left[\sqrt{-\delta}\Psi(X)\right]$
(3) When $\delta > 0,\ F[\Psi(X)] = \sqrt{\delta}\tan\left[\sqrt{\delta}\Psi(X)\right]$.

By balancing the linear term of the highest-order with the non-linear term in eq. (5), we get $m = n = 1$. Inserting eqs. (7) and (8) into eqs. (5)-(6), selecting the variable separation ansatz:

$$\Psi = p(x,t) + q(z) \tag{9}$$

and eliminating all the coefficients of polynomials of F, one gets a set of partial differential equations. Solving the set of differential equations simultaneously, we obtain the following results:

$$\delta = 0, \quad a_0 = b_0 = 0, \quad a_1 = p_x, \quad b_1 = p_t \tag{10}$$

and $p(x, t)$ satisfies $p = p[k(x + t)]$.

Inserting eqs. (10) and (9) into eqs. (7) and (4), taking the integration constant as $C(z, t)$, the corresponding variable separation solution of eq. (1) reads as:

$$H(x,z,t) = \frac{C(z,t)p_x}{(p+q)^2} \tag{11}$$

where $p = p(x + t)$, $q = q(z)$ and $C = C(z, t)$ are three arbitrary functions.

New coherent structures of eq. (1)

In this section, we will construct new coherent structures for the solution (11). Owing to the arbitrary functions $p(x, t)$, $q(z)$, and $C(z, t)$ involved in the solution (11), it is convenient to excite soliton structures. After attempts, we construct a new class of novel localized structures as follows.

(1) *Two solitons collision*

If we choose the arbitrary functions $p(x, t)$, $q(z)$ and $C(z, t)$ as:

$$p(x,t) = 1 + 0.2\mathrm{e}^{a(x+t)} + \mathrm{e}^{-b(x+t)},$$
$$q(z) = k\mathrm{e}^{cz^2},$$
$$C(z,t) = 2rz\mathrm{e}^{z^2+t^2}$$

respectively, the two solitons collision with parabolic motion can be obtained, see fig. 1(a).

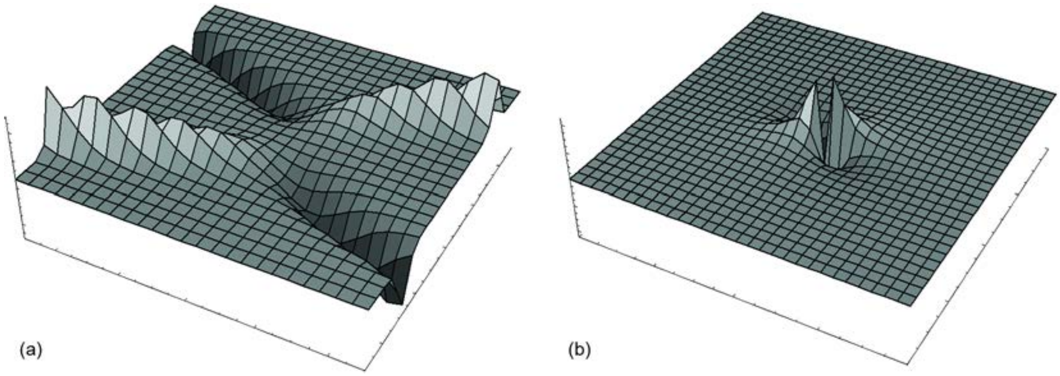

Figure 1. (a) Two solitons collision ($a = b = 2$, $c = 0.5$, $k = r = 1$, $t = 0$), (b) rogue wave ($a = 0.25$, $b = 0.125$, $c = 4$, $k = 0.5$, $t = 0$)

(2) *Rogue wave*

When $p(x, t)$, $q(z)$, and $C(z, t)$ possess the following forms:

$$p(x, t) = a(x + t)^2,$$
$$q(z) = b^2 + kz^2,$$
$$C(z, t) = cz + t^2$$

respectively, the rogue wave structure can be obtained and shown in fig. 1(b). The maximum amplitude of the rogue wave solution increases inversely with the parameter b.

(3) *Cross-like fractal soliton*

If we let:

$$p(x,t) = a(x+t)\ln(x+t)^2,$$
$$q(z) = r - bz\ln z^2,$$
$$C(z,t) = d + \ln(kz^2 + t^2)$$

respectively, the cross-like fractal structures can be obtained by setting the variable t and the parameters a, b, d, k, and r at special values as seen in fig. 2:

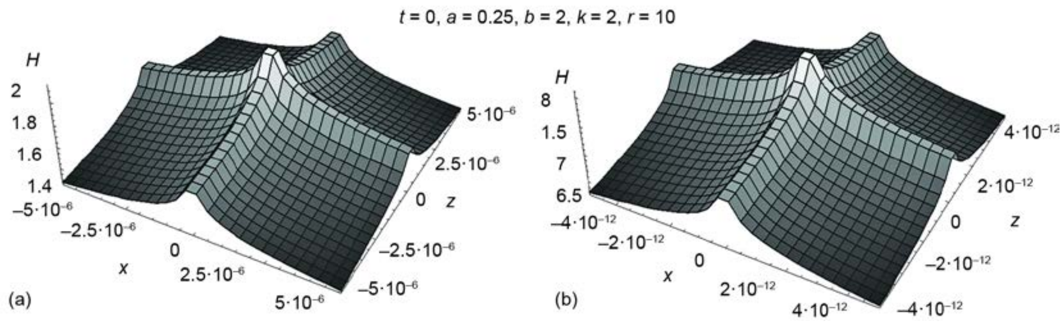

Figure 2. The cross-like fractal soliton

Figures 2(a) and 2(b) give the figures of the solution (11) with the settings blow, but x, and z in, respectively, $[-5\cdot 10^{-6}, -5\cdot 10^{-6}]$, $[-5\cdot 10^{-12}, -5\cdot 10^{-12}]$. The essential property of the fractal structures is the similarity of the figures in different axis scales. Figure 2 demonstrate that the cross-like fractal soliton holds its similarity in different ranges of x, and z.

(4) *Fusion interaction between three peakons*

Localized solutions might be helpful to the propagation processes for non-linear waves in (2+1)-D equations. Peakons are some types of weak solutions of the (2+1)-D equations, and there is a finite discontinuity of the first derivative in the wave peak [22, 23]. When $p(x, t)$, $q(z)$, and $C(z, t)$ are:

$$p(x,t) = 1 + b(x+t)\exp\left[a(x+t)\right], \qquad q(z) = 1 + kz^2 \tag{12}$$

$$c(z,t) = 0.1 + \mathrm{Sech}^2[r(z^2 - t^2)] \tag{13}$$

respectively, a fusion phenomenon between three peakons will be found, see fig. 3. Figure 3 shows an interaction phenomenon for three solitons. Two peakons are approaching to the third soliton along the z-axis direction fusing at the time $t = 0$, and then they leave along the z-axis, annihilate with the time increasing, their interaction is completely non-elastic.

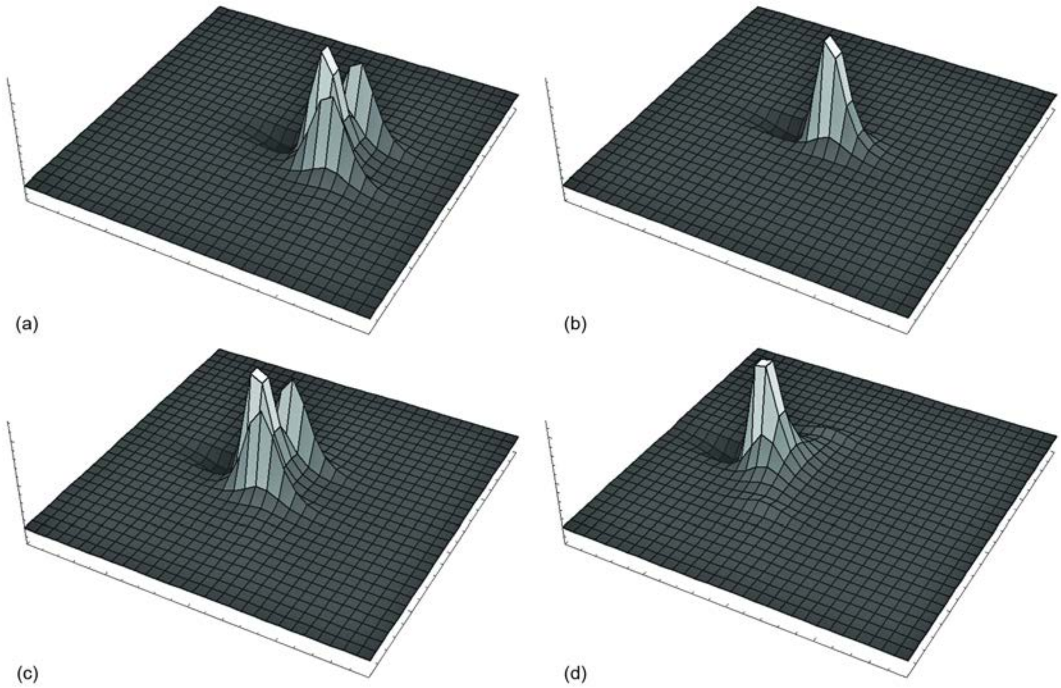

Figure 3. The annihilation plots with condition (12) and (13) at different times; (a) $t = -2$, (b) $t = 0$, (c) $t = 2$, and (d) $t = 4$

Conclusion

In this paper, we applied an improved mapping method and a variable separation hypothesis to the (2+1)-D SKdV equation and obtained a general variable separation solution with three arbitrary functions. Based on the general variable separation solution, abundant novel localized excitations, such as oscillating soliton, rogue wave and cross-like fractal structures have been constructed. The arbitrary functions in the obtained solutions imply that these solutions have rich spatial structures. It may be helpful in future studies for the intricate nature world. This method can be also extended to the other higher dimensional non-linear equations.

Acknowledgment

This work was supported by Natural Science Foundation of Yunnan Province under Grant No. 2013FZ113.

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Paper submitted: May 20, 2017
Paper revised: September 27, 2017
Paper accepted: September 29, 2017