# THE BARYCENTRIC RATIONAL INTERPOLATION COLLOCATION METHOD FOR BOUNDARY VALUE PROBLEMS

by

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*Higher-order boundary value problems have been widely studied in thermal science, though there are some analytical methods available for such problems, the barycentric rational interpolation collocation method is proved in this paper to be the most effective as shown in three examples.*

Key words: *barycentric rational interpolation, interpolation nodes higher-order boundary value problems*

## Introduction

The interpolation collocation method as a numerical method to solve differential equations has many merits, such as simple calculation and convenient program [1-4]. In numerical analysis, interpolation nodes sometimes are not free to be chosen, especially for the interpolation problems with equidistant nodes, for such case Lagrange polynomial interpolation is unstable, but the piecewise polynomial interpolation can be used, such as the spline interpolation, it can overcome the numerical instability [5]. Sometimes the precision of rational function interpolation is higher than that of the polynomial interpolation, and it can effectively overcome the instability problems. In a rational function space $R_{m,n}$, its elements are the rational functions of the polynomials. The following theorem shows that the rational function interpolation problem must have a solution [6].

*Theorem 1* [6]. Suppose $\{(x_j, f_j),\ j = 0, 1,\ldots, n)\}$ are $n + 1$ real number pairs, $x_j$ is different from each other. $\{u_j, j = 0, 1,\ldots , n\}$ are $n + 1$ real numbers, we have the following.

(1) If $u_k \neq 0$, there has rational function $r(x) \in R_{m,n}$:

$$r(x) = \frac{\sum_{j=0}^{n} \frac{u_j}{x - x_j} f_j}{\sum_{j=0}^{n} \frac{u_j}{x - x_j}}$$

the interpolation of $r(x)$ at $x_k$ is $f_k$ , *i. e.* $\lim_{x \to k_k} r(x) = f_k$.

(2) Otherwise, the rational function of any interpolation real number pairs, $\{(x_j, f_j), j = 0, 1, \ldots, n)\}$, can be expressed as barycentric interpolation.

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In this paper, we select suitable points as interpolation nodes, use barycentric rational interpolation collocation method (BRICM) [7-9] to solve higher-order boundary value problems (BVP) in the form:

$$\begin{cases} \sum_{i=0}^{m} a_i(x)u^{(i)}(x) = f(x), \quad a \le x \le b \\ u^{(i)}(a) = c_i, \quad u^{(j)}(b) = d_i, \quad 0 \le i, \quad j \le m-1 \end{cases} \tag{1}$$

The barycentric rational collocation method has been proved to be effective to solve non-linear high-dimensional Fredholm integral equations of the second kind [10] and non-linear parabolic partial differential equations [11].

## Barycentric rational interpolation collocation method

According to the ideas of the collocation method, all-order derivatives of function at nodes can be approximated as linear weighted sum of the function value at nodes. Consider the function $u(x)$ which is defined in interval [0, 1], the values of function $u(x)$ at nodes are $u_i = u(x_i)$, $i = 1, 2, \ldots, n$, and all-order derivatives of $u(x)$ at the nodes can be expressed as the linear weighted sum of the function value [7-9]:

$$u_i^{(m)} = u^{(m)}(x_i) = \frac{\mathrm{d}^m u(x_i)}{\mathrm{d}x^m} = \sum_{j=1}^{n} D_{ij}^{(m)} u_j, \quad m = 1, 2, \ldots n \tag{2}$$

Written in matrix form is $u^{(m)} = D^{(m)}u$, where $u^{(m)} = [u_1^{(m)}, u_2^{(m)}, \ldots, u_n^{(m)}]^T$ is the column vector of the $m$ order derivatives of unknown function at nodes, $D^{(m)}$ is the $m$ order differential matrix of unknown function, the elements of $D_{ij}^{(m)}$ are the weighted coefficient.

Barycentric interpolation primary function is denoted by $L_j(x)$, the barycentric interpolation $u(x)$ can be expressed as:

$$u(x) = \sum_{j=1}^{n} L_j(x) u_j$$

So, the first order and second order derivatives of $u(x)$ can be expressed, respectively, as:

$$u'(x) = \sum_{j=1}^{n} L_j'(x) u_j$$

$$u''(x) = \sum_{j=1}^{n} L_j''(x) u_j$$

Barycentric interpolation primary function is:

$$L_j(x) = \frac{\dfrac{w_j}{x - x_j}}{\displaystyle\sum_{k=1}^{n} \frac{w_k}{x - x_k}} \tag{3}$$

where $w_j$ is barycentric interpolation weight, defined as:

$$w_j = \sum_{i\in J_k} (-1)^i \prod_{j=i,\, j\neq k}^{i+d} \frac{1}{x_k - x_j}$$

The index set $J_k = \{ i \in I,\ k - d \leq i \leq k\}$ is based on the distribution of interpolation nodes and the choice of $d$. Multiplying $x - x_i(i \neq j)$ at both sides of (3), we obtain:

$$L_j(x)\sum_{k=1}^{n} w_k \frac{x - x_i}{x - x_k} = w_j \frac{x - x_i}{x - x_j} \tag{4}$$

For simplicity, we introduce a new variable $s(x)$ defined as:

$$s(x) = \sum_{k=1}^{n} w_k \frac{x - x_i}{x - x_k}$$

which meets the following properties:

$$s(x_i) = w_i\,, \quad s'(x) = \sum_{k\neq i} \frac{w_k}{x_i - x_k}\,, \quad s''(x) = -2\sum_{k\neq i} \frac{w_k}{(x_i - x_k)^2}$$

Calculating derivative with respect to $x$ at both sides of eq. (4), we get:

$$L_j'(x)s(x) + L_j(x)s'(x) = w_j \left(\frac{x - x_i}{x - x_j}\right)'$$

$$L_j''(x)s(x) + 2L_j'(x)s'(x) + L_j(x)s''(x) = w_j \left(\frac{x - x_i}{x - x_j}\right)''$$

we know $L_j(x_i) = 0(j \neq i)$,so we can get:

$$L_j'(x_i) = \frac{\dfrac{w_j}{w_i}}{x_i - x_j}\ (j \neq i)\,, \quad L_j''(x_i) = -2\frac{\dfrac{w_j}{w_i}}{x_i - x_j}\left(\sum_{k\neq i} \frac{\dfrac{w_k}{w_i}}{x_i - x_k} + \frac{1}{x_i - x_j}\right)(j \neq i)$$

If $i = j$, $\Sigma_{j=1}^{n} L_j(x) = 1$. Calculating derivative with respect to $x$ at both sides, we get $\Sigma_{j=1}^{n} L_j^{(m)}(x) = 0$. So, $L_j'(x_i) = -\Sigma_{j=1} L_j'(x_i)$, $L_j''(x_i) = -\Sigma_{j\neq i} L_j''(x_i)$, and the differential matrix of barycentric interpolation is [2]:

$$D_{ij}^{(1)} = L_j'(x_i)\,, \quad D_{ij}^{(2)} = L_j''(x_i)$$

$$\begin{cases} D_{ij}^{(m)} = m\left(D_{ii}^{(m-1)} D_{ij}^{(1)} - \dfrac{D_{ij}^{(m-1)}}{x_i - x_j}\right), & i \neq j \\ D_{ii}^{(m)} = -\displaystyle\sum_{j=1,\, j\neq i}^{n} D_{ij}^{(m)} \end{cases} \tag{5}$$

In view of eq. (1), let interval [0, 1] be dispersed as $0 = x_1 < x_2 < \ldots < x_n = 1$, let $u_1$, $u_2$, …, $u_n$ as the values of function $u(x)$ at disperse nodes $x_1$, $x_2$, …, $x_n$, using the barycentric rational interpolation collocation can get approximate function $u(x)$:

$$u(x) = \sum_{j=1}^{n} L_j(x) u_j \tag{6}$$

Taking eq. (6) into eq. (1), we can get:

$$\sum_{j=1}^{n}\sum_{i=0}^{m} a_i(x) L_j^{(i)}(x) u_j = f(x) \tag{7}$$

Let eq. (7) accurately be established at disperse nodes, we can get $n$ equations:

$$\sum_{j=1}^{n}\sum_{i=0}^{m} a_i(x) L_j^{(i)}(x_k) u_j = f(x), \quad k = 1, 2, ..., n \tag{8}$$

*i. e.*

$$\sum_{j=1}^{n}\sum_{i=0}^{m} a_i(x) D_{kj}^{(i)}(x_k) u_j = f(x), \quad k = 1, 2, ..., n \tag{9}$$

Write eq. (9) in matrix form:

$$LU = F \tag{10}$$

There

$$L = A_i D^{(i)}, \quad U = [u_1, u_2, ..., u_n]^T$$

$$A_i = \text{diag}[a_i(x)], \quad F = [f_1, f_2, ..., f_n]$$

Taking eq. (6) into initial conditions, we have:

$$\sum_{j=1}^{n} D_{1j}^{(i)} u_j = c_i, \quad \sum_{k=1}^{n} D_{nk}^{(j)} u_k = d_j, \quad 0 \le i, \quad j \le m-1 \tag{11}$$

On the basis of ordinary differential equation theory (Picard iterative method), $\forall f(x) \in C[a, b]$, eq. (1) has unique solution in $[a, b]$.

*Theorem 2:* Suppose $u(x)$ is the exact solution of eq. (1), if $u_n(x)$ is the numerical solution of eq. (1), we have $Lu_n(x_k) = f(x_k)$ ($k$ = 1, 2,…, $n$), and $\lim_{n\to\infty} u_n(x) = u(x)$.

The proof is straightforward and it is not written here for simplicity.

## Numerical experiment

In this section, three numerical examples are studied to demonstrate the accuracy of the present method.

*Example 1.* Consider the following fifth-order boundary value problem [2, 3]:

$$\begin{cases} u^{(5)}(x) - u(x) = -15e^x - 10xe^x, \quad 0 \le x \le 1 \\ u(0) = 0, \quad u'(0) = 1, \quad u''(0) = 0, \\ u(1) = 0, \quad u'(1) = -\text{e} \end{cases}$$

The exact solution is $u_T(x) = (1 - x)e^x$. The numerical results are shown in tab. 1 and fig. 1.

**Table 1. Numerical results for *Example 1***

| x | $u_T(x)$ | Present method | Decomposition method [2] | B-spline [2] | Variational iteration [3] |
|---|---|---|---|---|---|
| x | $u_T(x)$ | Absolute error | Absolute error | Absolute error | Absolute error |
| 0 | 0.0000 | $2.1\cdot10^{-12}$ | 0.000 | 0.000 | 0.000 |
| 0.1 | 0.0995 | $5.6\cdot10^{-12}$ | $3\cdot10^{-11}$ | $8\cdot10^{-3}$ | $1.077\cdot10^{-4}$ |
| 0.2 | 0.1954 | $3.4\cdot10^{-11}$ | $2\cdot10^{-10}$ | $1.2\cdot10^{-3}$ | $1.077\cdot10^{-4}$ |
| 0.3 | 0.2835 | $6.7\cdot10^{-11}$ | $4\cdot10^{-10}$ | $5\cdot10^{-3}$ | $2.477\cdot10^{-4}$ |
| 0.4 | 0.3580 | $9.3\cdot10^{-11}$ | $8\cdot10^{-10}$ | $3\cdot10^{-3}$ | $3.729\cdot10^{-4}$ |
| 0.5 | 0.4122 | $1.0\cdot10^{-10}$ | $1.2\cdot10^{-9}$ | $8\cdot10^{-3}$ | $4.202\cdot10^{-4}$ |
| 0.6 | 0.4373 | $9.8\cdot10^{-11}$ | $2\cdot10^{-9}$ | $6\cdot10^{-3}$ | $3.643\cdot10^{-4}$ |
| 0.7 | 0.4229 | $7.7\cdot10^{-11}$ | $2.2\cdot10^{-9}$ | 0 | $2.364\cdot10^{-4}$ |
| 0.8 | 0.3561 | $4.5\cdot10^{-11}$ | $1.9\cdot10^{-9}$ | $9\cdot10^{-3}$ | $1.158\cdot10^{-4}$ |
| 0.9 | 0.2214 | $1.5\cdot10^{-11}$ | $1.4\cdot10^{-9}$ | $9\cdot10^{-3}$ | $8.760\cdot10^{-5}$ |
| 1.0 | 0 | 0 | 0.000 | 0.000 | 0.0000 |

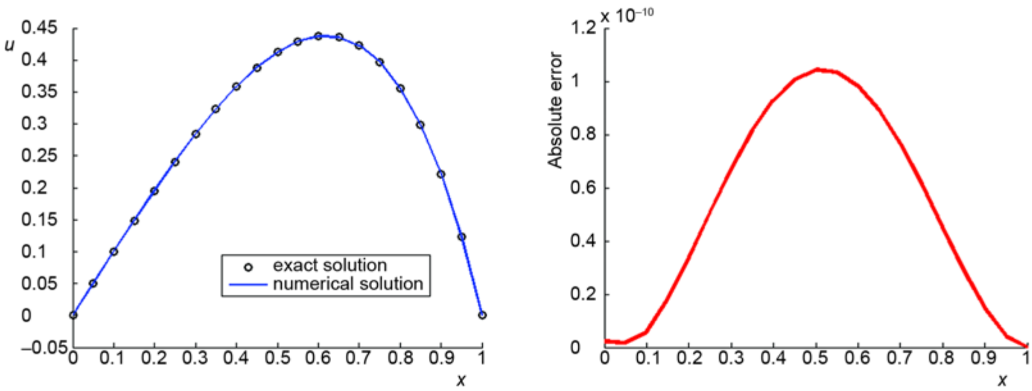


**Figure 1. Present numerical method for *Example 1*, the left picture is the exact solution and numerical solution, the right picture is the absolute error**

*Example 2.* Consider the eighth-order boundary value problem [4]:

$$\begin{cases} u^{(8)}(x) - u(x) = -8e^x, \quad 0 \le x \le 1 \\ u(0) = 1, \quad u'(0) = 0, \quad u''(0) = -1, \\ u'''(0) = -2, \quad u^{(4)}(0) = -3, \quad u^{(5)}(0) = -4, \\ u^{(6)}(0) = -5, \quad u^{(7)}(0) = -6 \end{cases}$$

The exact solution is $u_T(x) = (1 - x)e^x$. The numerical results are shown in tab. 2 and fig. 2.

**Table 2. Numerical results for *Example 2***

| x | $u_T(x)$ | Present method | Present method | Method in [4] |
|---|---|---|---|---|
| x | $u_T(x)$ | $u_{21}(x)$ | Absolute error | Absolute error |
| 0.25 | 0.9630 | 0.9630 | $1.4\cdot10^{-10}$ | $1.0\cdot10^{-7}$ |
| 0.5 | 0.8244 | 0.8244 | $8.4\cdot10^{-9}$ | $3.1\cdot10^{-6}$ |
| 0.75 | 0.5293 | 0.5293 | $3.2\cdot10^{-8}$ | $5.5\cdot10^{-5}$ |
| 1.0 | 0 | 0 | $2.3\cdot10^{-8}$ | $4.2\cdot10^{-4}$ |

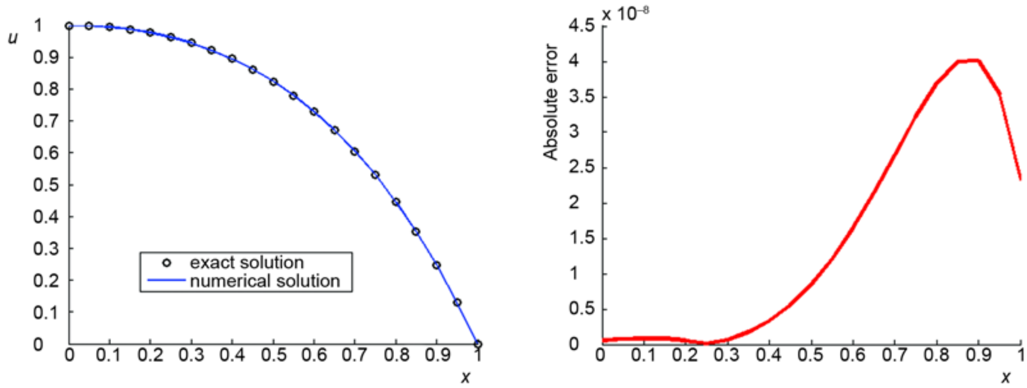


**Figure 2. Present numerical method for *Example 2*, the left picture is the exact solution and numerical solution, the right picture is the absolute error**

*Example 3.* Consider the eighth-order boundary value problem [4].

$$\begin{cases} u^{(8)}(x)-u(x)=-8e^{x}, \quad 0\le x\le 1 \\ u(0)=1, \quad u''(0)=-1, \quad u^{(4)}(0)=-3, \\ u^{(6)}(0)=-5, \quad u(1)=0, \quad u''(1)=-2e, \\ u^{(4)}(1)=-4e, \quad u^{(6)}(1)=-6e \end{cases}$$

The exact solution is $u_T(x)=(1-x)e^{x}$. The numerical results are shown in tab. 3 and fig. 3.

**Table 3. Numerical results for *Example 3***

| x | $u_T(x)$ | Present method | Present method | Modified decomposition method [4] |
|---|---|---|---|---|
| x | $u_T(x)$ | $u_{21}(x)$ | Absolute error | Absolute error |
| 0.25 | 0.9630 | 0.9630 | $2.1\cdot10^{-7}$ | $7.27\cdot10^{-5}$ |
| 0.5 | 0.8244 | 0.8244 | $2.9\cdot10^{-7}$ | $1.025\cdot10^{-4}$ |
| 0.75 | 0.5293 | 0.5293 | $2.0\cdot10^{-7}$ | $7.24\cdot10^{-5}$ |

## Conclusions and remarks

In this paper, we use the barycentric rational interpolation collocation method [7-9] to solve higher-order BVP. The numerical results demonstrate that the method is quite accurate and efficient for linear higher-order BVP. It is worthy to note that this method can be generalized to higher order BVP. All computations can be performed using some mathematical software, making the method much more attractive.

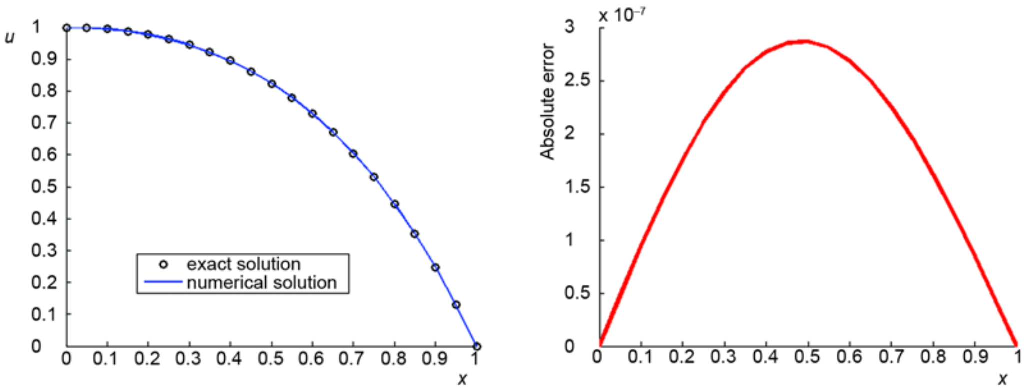


**Figure 3. Present numerical method for *Example 3*, the left picture is the exact solution and numerical solution, the right picture is the absolute error**

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