

## MONTE CARLO METHOD WITH CONTROL VARIATE FOR INTEGRAL EQUATIONS

by

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*This paper combines the successive substitution method and Monte Carlo method with a control variate to solve Fredholm integral equations of the second kind. Some examples are given to elucidate the solution process and the results reveal the efficiency of the method.*

**Key words:** Monte Carlo method, control variate, Fredholm integral equation

### Introduction

Integral equations are often involved in the mathematical formulation of physical phenomena, they can be encountered in various fields of science, such as the heat radiation problems. Fredholm integral equations for 1-D backward heat conduction problems were studied in [1]. There are many methods to obtain the solution of an integral equation, such as the meshless method [2], the Legendre-Gauss-Lobatto collocation method [3], the piecewise linear maximum entropy method [4], *etc.* In this paper, it is our intention to numerically solve the Fredholm integral equation of the second kind by combining the classical Monte Carlo method and the Monte Carlo method with a control variate [5-8].

In this paper, we will consider the following Fredholm integral equation of the second kind:

$$\phi(x) = g(x) + \lambda \int_0^1 k(x,t)\phi(t)dt, \quad 0 \leq x \leq 1 \quad (1)$$

where the function  $g(x) \in L_2[0, 1]$ , and the kernel  $k(x, t)$  is measurable in the square  $0 \leq x, t \leq 1$  and  $\int_0^1 \int_0^1 |k(x,t)|^2 dxdt < \infty$ ,  $\lambda$  – a parameter, and  $\phi(x) \in L_2[0, 1]$  – an unknown function to be determined.

### Overview of the method

#### *The successive approximations method*

The successive approximations method introduces the recurrence relation:

$$\phi_n(x) = g(x) + \lambda \int_0^1 k(x,t)\phi_{n-1}(t)dt, \quad n \geq 1 \quad (2)$$

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where the zeroth approximation  $\phi_0$  can be any selective real valued function, we always start with an initial guess for  $\phi_0$ , and by using eq. (2), several successive approximations  $\phi_k$ ,  $k \geq 1$  will be determined.

For eq. (1), we take the initial guess:

$$\phi_0(x) = g(x) \quad (3)$$

by using eq. (2), we have:

$$\phi_1(x) = g(x) + \lambda \int_0^1 k(x,t)\phi_0(t)dt \quad (4)$$

$$\phi_2(x) = g(x) + \lambda \int_0^1 k(x,t)\phi_1(t)dt \quad (5)$$

⋮

$$\phi_m(x) = g(x) + \lambda \int_0^1 k(x,t)\phi_{m-1}(t)dt \quad (6)$$

using an integral operator,  $\phi_1(x)$  can be written:

$$\phi_1 = g + \lambda K \phi_0 = g + \lambda K g \quad (7)$$

in the same way, we obtain:

$$\phi_2 = g + \lambda K \phi_1 = g + \lambda K g + \lambda^2 K^2 g \quad (8)$$

⋮

$$\phi_m = g + \lambda K \phi_{m-1} = g + \lambda K g + \lambda^2 K^2 g + \dots + \lambda^m K^m g \quad (9)$$

the question of convergence of  $\phi_m$  is justified by noting the following theorem.

*Theorem 1.* For continuous functions  $g(x)$  and the kernel  $k(x, t)$  in eq. (1), if the condition  $|\lambda| < 1/A$  holds, where  $\sup_{(x, t) \in [0, 1]^2} |k(x, t)| = A$ , then the sequence of successive approximations  $\phi_m(x)$ ,  $m \geq 0$  converges to the solution  $\phi(x)$  of the integral eq. (1) and the solution  $\phi(x)$  can be written:

$$\phi(x) = g + \lambda K g + \lambda^2 K^2 g + \dots + \lambda^m K^m g + \dots \quad (10)$$

*Proof* [9].

By the expression of successive approximations (10), in order to obtain the approximations solution of the integral eq. (1), the  $n^{\text{th}}$  approximation  $\phi_n(x)$  ( $n \geq 1$ ) of  $\phi(x)$  will be computed by means of calculating the following  $n$ -dimensional integration, namely:

$$\underbrace{\int_0^1 \int_0^1 \dots \int_0^1}_{n} k(x, t_1) k(t_1, t_2) \dots k(t_{n-1}, t_n) g(t_n) dt_1 dt_2 \dots dt_n \quad (11)$$

### Classical Monte Carlo method

The classical Monte Carlo method for computing the integral:

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1, x_2, \dots, x_m) dx_1 dx_2 \cdots dx_m \quad (12)$$

is to represent it as an expected value of some random variable. Let us write the integral:

$$I = E \frac{f(x)}{h_X(x)} \quad (13)$$

where  $x = (x_1, x_2, \dots, x_m)$ , and  $h_X(x)$  is any probability density function. For simplicity, assume:

$$h_X(x) = \begin{cases} 1, & \text{if } 0 < x_i < 1, \quad i = 1, 2, \dots, m \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

consequently:

$$I = E[f(x)] \quad (15)$$

in classical Monte Carlo method, the approximation of the expectation  $E[f(x)]$  is given by:

$$E[f(x)] \approx \theta_{MC} = \frac{1}{N} \sum_{i=1}^N f(x_i), \quad x = (x_1, x_2, \dots, x_m) \quad (16)$$

the variance of  $\theta_{MC}$  is calculated:

$$\text{Var}(\theta_{MC}) = \frac{1}{N} \left[ \int_0^1 \int_0^1 \cdots \int_0^1 f^2(x_1, x_2, \dots, x_m) dx_1 dx_2 \cdots dx_m - I^2 \right] \quad (17)$$

Hence the precision of the usual Monte Carlo method is  $O(N^{-1/2})$ .

### Monte Carlo method with a control variate

Now we want to estimate:

$$\theta = \int_I f(x) dx = E_{PX} \frac{f(X)}{p_X(X)} \triangleq E_{PX} [f^*(X)]$$

where  $p_X(x) (x \in I)$  is a probability density of the random variable  $X$ . For any constant  $\beta$ , the quantity:

$$\bar{\theta}_{MC} = \frac{1}{N} \sum_{i=1}^N (f^*(X_i) + \beta \{\xi(X_i) - E[\xi(X)]\}) \quad (18)$$

is an unbiased estimator of  $\theta$ . To determine the best value of  $\beta$ , note that:

$$\begin{aligned} \text{Var}(f^*(X_i) + \beta \{\xi(X_i) - E[\xi(X)]\}) &= \\ &= \text{Var}[f^*(X_i)] + \beta^2 \text{Var}[\xi(X)] + 2\beta \text{Cov}[f^*(X), \xi(X)] \end{aligned} \quad (19)$$

Equation (19) is minimized when  $\beta = \beta^*$ , where:

$$\beta^* = -\frac{\text{Cov}[f^*(X), \zeta(X)]}{\text{Var}[X_i]} \quad (20)$$

and for this value, the variance of  $\bar{\theta}_{\text{MC}^*}$  is:

$$\begin{aligned} \text{Var}(\bar{\theta}_{\text{MC}^*}) &= \frac{1}{N} \sum_{i=1}^N (f^*(X_i) + \beta^* \{\zeta(X_i) - E[\zeta(X)]\})^2 \\ &= \frac{\text{Var}[f^*(X)]}{N} - \frac{\{\text{Cov}[f^*(X), \zeta(X)]\}^2}{N\text{Var}[X_i]} \end{aligned} \quad (21)$$

the function  $\zeta(X)$  is called a control variate for  $f^*(x)$ .

Let  $T_n(x)$  be the Taylor's expansion of the function  $f(x)$  at point  $EX$ , order  $n$ , *i. e.* for  $x \in I$ :

$$\begin{aligned} T_n(x) &= f^*(EX) + f^{*'}(EX)(x - EX) + \frac{1}{2!} f^{*''}(EX)(x - EX)^2 + \\ &\quad + \dots + \frac{1}{n!} f^{*(n)}(EX)(x - EX)^n \end{aligned} \quad (22)$$

on the total defined interval  $I$  of the target function  $f^*(x)$ , let the control variate  $\zeta(x) = T_n(x)$ , then eq. (18) is expressed by:

$$\bar{\theta}_{\text{MC}^*} = \frac{1}{N} \sum_{i=1}^N (f^*(X_i) + \beta^* \{T_n(X_i) - E[T_n(X)]\}) \quad (23)$$

By eq. (22), we can see that, the more difference between the values  $x$  and  $EX$ , the more error exists when  $T_n(x)$  is used to approximate the target function  $f^*(x)$ , motivated by this, we propose the Taylor's formula of  $f^*(x)$  at midpoint of each subinterval  $I_i$ , where  $I = I_1 \cup I_2 \cup \dots \cup I_L$ ,  $I_i \cap I_j = \emptyset$ ,  $i \neq j = 1, 2, \dots, L$ . For simplicity, if one draws independent and identically distributed random variable uniformly from interval  $[0, 1]$ , namely:

$$P_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (24)$$

Now consider a uniform partition of the closed  $[0, 1]$  given by  $I_i = [(i-1)/L, i/L]$ ,  $i = 1, 2, \dots, L$ , and the midpoints of each subinterval  $I_i$  are written:

$$EX_i = \frac{i}{L} - \frac{1}{2L}, \quad i = 1, 2, \dots, L \quad (25)$$

let  $T_n^*(x)$  be Taylor's expansion of the function  $f^*(X)$  at point  $EX_i$  of order  $n$  based on subinterval  $I_i = (i = 1, 2, \dots, L)$ , *i. e.*:

$$T_n^*(x) = f^*(EX_i) + f^{*'}(EX_i)(x - EX_i) + \frac{1}{2!} f^{*''}(EX_i)(x - EX_i)^2 + \dots + \frac{1}{n!} f^{*(n)}(EX_i)(x - EX_i)^n, \quad x \in I_i \tag{26}$$

then eq. (23) becomes:

$$\bar{\theta}_{MC^*} = \frac{1}{N} \sum_{i=1}^N \left( f^*(X_i) + \beta^* \left\{ T_n^*(X_i) - E[T_n^*(X)] \right\} \right) \tag{27}$$

where  $X$  is a random variable with probability density  $p_X(x)$ .

### Discussion of the numerical experiments

In this section, we are going to demonstrate some numerical result for determine  $\phi(x)$  in the integral eq. (1). All the computations are performed on the PC. To give a clear overview of the present method, the following examples will be considered and the solution of which is to be obtained.

Here  $N$  denotes the number of samples, MC represents classical Monte Carlo method simulation results, and  $MC^*$  represents the results for present method with one control variate  $T_2$ , that is eq. (24) with  $\beta^* = 1$ .

*Example 1.* Consider the following equation:

$$\phi(x) = \cos(2\pi x) + \frac{1}{2} \sin(4\pi x) - \int_0^1 \sin(4\pi x + 2\pi t) \phi(t) dt, \quad 0 \leq x \leq 1 \tag{28}$$

for which the exact solution is  $\phi(x) = \cos(2\pi x)$ .

The results obtained for  $u(x)$  with MC and  $MC^*$  are presented in tabs. 1 and 2 and fig. 1. The error of Monte Carlo simulation for *Example 1* are presented in fig. 2.

**Table 1.**  $N = 1000, L = 20$  for  $u(x)$  with MC

$x$	MC	Exact-MC	Exact
0.1	0.82658	1.8e-2	0.80902
0.2	0.31454	5.5e-3	0.30902
0.3	-0.30920	1.8e-4	-0.30902
0.4	-0.81855	9.5e-3	-0.80902
0.5	-1.01509	1.5e-2	-1.00000
0.6	-0.79085	1.8e-2	-0.80902
0.7	-0.30276	6.3e-3	-0.30902
0.8	0.29076	1.8e-2	0.30902
0.9	0.78879	2.0e-2	0.80902
1.0	0.98332	1.7e-2	1.00000

**Table 2.**  $N = 1000, L = 20$  for  $u(x)$  with  $MC^*$

$x$	$MC^*$	Exact- $MC^*$	Exact
0.1	0.80946	4.4e-4	0.80902
0.2	0.30889	1.3e-4	0.30902
0.3	-0.30943	4.1e-4	-0.30902
0.4	-0.80900	2.0e-5	-0.80902
0.5	-1.00022	2.2e-4	-1.00000
0.6	-0.80887	1.5e-4	-0.80902
0.7	-0.30894	8.0e-5	-0.30902
0.8	0.30893	9.0e-5	0.30902
0.9	0.80915	1.3e-4	0.80902
1.0	1.00011	1.1e-4	1.00000

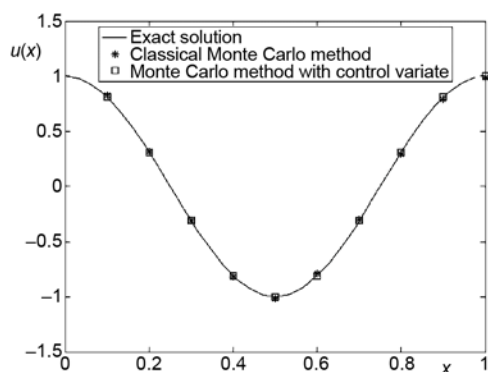


Figure 1. The solution of Example 1 with MC and MC\*

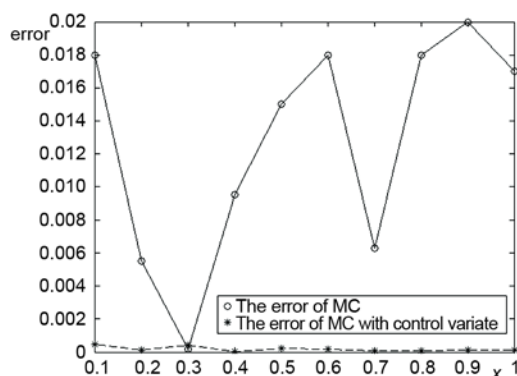


Figure 2. The error of Monte Carlo simulation for Example 1

Example 2. Consider the following equation:

$$\phi(x) = 0.9x^2 + 0.5 \int_0^1 x^2 t^2 \phi(t) dt, \quad 0 \leq x \leq 1 \quad (29)$$

for which the exact solution is  $\phi(x) = x^2$ .

The results obtained for  $u(x)$  with MC and MC\* are presented in tabs. 3 and 4 and fig. 3. The error of Monte Carlo simulation for Example 2 are presented in fig. 4.

Table 3.  $N = 1000, L = 40$  for  $u(x)$  with MC

$x$	$M^*$	Exact-MC	Exact
0.1	0.01001	1.0e-5	0.01
0.2	0.04012	1.2e-4	0.04
0.3	0.08956	4.4e-4	0.09
0.4	0.15966	3.4e-4	0.16
0.5	0.24958	4.2e-4	0.25
0.6	0.35916	8.4e-4	0.36
0.7	0.48957	4.3e-4	0.49
0.8	0.64176	1.8e-3	0.64
0.9	0.80798	2.0e-3	0.81
1.0	1.00352	3.5e-3	1.00

Table 4.  $N = 1000, L = 40$  for  $u(x)$  with MC\*

$x$	MC*	Exact-MC*	Exact
0.1	0.00999	1.0e-5	0.01
0.2	0.03997	3.0e-5	0.04
0.3	0.08993	7.0e-5	0.09
0.4	0.15987	1.3e-4	0.16
0.5	0.24980	2.0e-4	0.25
0.6	0.35971	2.9e-4	0.36
0.7	0.48960	4.0e-4	0.49
0.8	0.63948	5.2e-4	0.64
0.9	0.80934	6.6e-4	0.81
1.0	0.99905	9.5e-4	1.00

**Conclusion**

The present study, successfully applies the successive approximations method and the Monte Carlo method with control variate to solve Fredholm integral equations of the second kind. From the numerical examples, it can be seen that the Monte Carlo method based

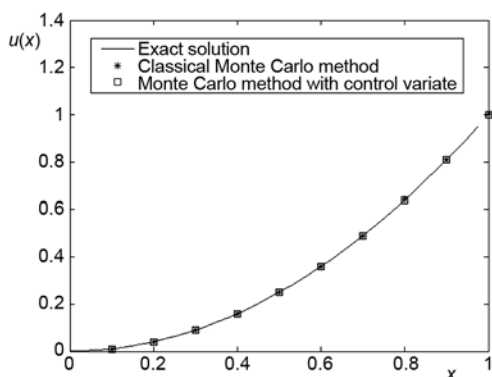


Figure 3. The solution of Example 2 with MC and MC\*

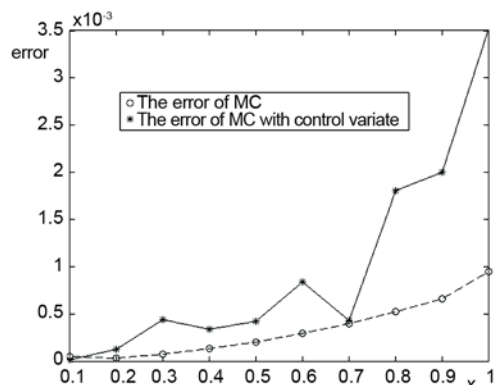


Figure 4. The error of Monte Carlo simulation for Example 2

on control variate is more accurate than classical Monte Carlo method to estimate the solution of Fredholm integral equations.

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