THE FRACTIONAL POWER SERIES METHOD
An Efficient Candidate for Solving Fractional Systems

by

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The fractional power series method was originally proposed to solve a fractional differential equation. This paper extends the method to a system of fractional differential equations with great success. How to construct an initial solution, plays an important role in the solution process and an example is given to elucidate the choice of the initial solution.

Key words: *approximate solution, the system of fractional differential equations, Caputo fractional derivative, fractional power series method*

Introduction

The solutions of system of fractional differential equations are much involved. In general, there exists no method that yields exact solutions for the system. In recent years, many researchers have focused on the approximate analytical solutions of the system of fractional differential equations and some methods have been developed such as Adomian decomposition method [1], homotopy perturbation method [2], variational iteration method [3], homotopy analysis method [4], Taylor series method [5], and Daftardar-Gejji-Jafaris method [6], *etc.*

Recently, the fractional power series method (FPSM) has been proposed by El-Ajou *et al*. [7]. The method is one of the most useful techniques to solve linear system and non-linear system of fractional differential equations with a fast convergence rate and small calculation error. Another important advantage is that this method can be used directly to non-linear system of fractional PDE without requiring linearization, discretization, Adomian polynomials *etc.*

In this paper, we will study the following systems of fractional PDE:

$$\mathrm{D}_t^{\alpha_i} u_i(x,t) = f_i(x,t;u_1,u_2;u_{1x},u_{2x}) \tag{1}$$

subject to the initial conditions:

$$u_i(x,0) = \phi_i(x) \tag{2}$$

where the fractional derivative is considered in the Caputo sense, and α_i are parameters describing the order of the Caputo fractional derivative ($0 < \alpha_i \le 1$), f_i, ϕ_i are given functions, $i = 1, 2$.

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Such systems apply to description of many phenomena in thermal physics, viscoelasticity, chemistry, biology, and engineering [8-15].

Purpose of this paper is to use the FPSM to provide a reliable algorithm for the solutions to the systems of fractional differential equations. Two typical cases are considered, which are often used in applications.

Basic definitions of fractional calculus

In order to proceed, we need the following basic definitions and properties of fractional calculus theory which shall be used in this paper. See [16] for detail of proof.

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

The Riemann-Liouville fractional integral operator is defined as follows.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_\mu$, $\mu \geq -1$ is defined:

$$\mathrm{J}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\mathrm{d}s$$

$$\mathrm{J}^0 f(x) = f(x)$$

Properties of the operator J^{α} can be found in [16] and we mention only the following. For $\alpha, \beta \geq 0$, $x > 0$, and $\gamma > -1$:

$$\mathrm{J}^{\alpha}\mathrm{J}^{\beta} f(x) = \mathrm{J}^{\alpha+\beta} f(x)$$

$$\mathrm{J}^{\alpha}\mathrm{J}^{\beta} f(x) = \mathrm{J}^{\beta}\mathrm{J}^{\alpha} f(x)$$

$$\mathrm{J}^{\alpha}(x^{\gamma}) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha+\gamma)} x^{\gamma+\alpha}$$

The Riemann-Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce the fractional differential operator, D^{α}, proposed by Caputo in his work on the theory of viscoelasticity [17]. For more information about Caputo definition and properties see [16].

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined:

$$\mathrm{D}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s$$

for $m - 1 < \alpha \leq m$, $m \in N^+$, $x > 0$ and $f \in C_{-1}^m$.

We recall here two of its basic properties:

$$\mathrm{D}^{\alpha}\mathrm{J}^{\alpha} f(x) = f(x)$$

$$\mathrm{J}^{\alpha}\mathrm{D}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!}, \qquad x > 0$$

Fractional power series

Power series is a fundamental tool in the study of elementary functions. They have been widely used in computational science for easily obtaining an approximation of functions. In thermal physics and many other sciences this power expansion has allowed scientist to make an approximate study of many differential equations. In this section, we will recall some important definitions and theorems of fractional power series theory [7].

Definition 1. A power series representation of the form:

$$\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} = c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots$$

where $0 \le m-1 < \alpha \le m$, $m \in N^+$ and $t \ge t_0$ is called a fractional power series about t_0, where t is a variable and c_n are the coefficients of the series.

Theorem 1. We have the following two cases for the $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \ge 0$:

(1) If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ converges when $t = b > 0$, then it converges whenever $0 \le t < b$,
(2) If $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ diverges when $t = d > 0$, then it diverges whenever $t > d$.

Theorem 2. For the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \ge 0$, there are only three possibilities:

(1) The series converges only when $t = 0$,
(2) The series converges for each $t ^3 \ge 0$,
(3) There is a positive real number, R, such that the series converges whenever $0 \le t < R$ and diverges whenever $t > R$.

Theorem 3. The series $\sum_{n=0}^{\infty} c_n t^{n}, -\infty < t < \infty$ has radius of convergence, R, if and only if the series $\sum_{n=0}^{\infty} c_n t^{n\alpha}, t \ge 0$ has radius of convergence $R^{1/\alpha}$.

The following property plays an important role in next section.

Theorem 4. Suppose that the fractional power series $\sum_{n=0}^{\infty} c_n t^{n\alpha}$ has radius of convergence $R > 0$. If $f(t)$ is a function defined by $f(t) = \sum_{n=0}^{\infty} c_n t^{n\alpha}$ on $0 \le t < R$, then for $m-1 < \alpha \le m$ and $0 \le t < R$, we have:

$$\mathrm{D}^{\alpha} f(t) = \sum_{n=0}^{\infty} c_n \frac{\Gamma(n\alpha+1)}{\Gamma\left[(n-1)\alpha+1\right]} t^{(n-1)\alpha}$$

Solving the system of eqs. (1) and (2) by FPSM

To solve the system by FPSM, the form of the solutions must be chosen appropriately, which is the key technique of the algorithm.

Firstly, we have observed that, in general, the functions having the forms:

$$u_i(x,t) = \sum_{k=0}^{\infty} a_{ik}(x) t^{\alpha_i k} \quad (i = 1, 2)$$

are not the solutions of the systems of eqs. (1) and (2).

We explain the fact by considering the following problem:

$$\begin{cases} \mathrm{D}_t^{0.5} u_1(x,t) = u_{2x} - u_2 - u_1 \\ \mathrm{D}_t^{0.2} u_2(x,t) = u_{1x} - u_2 - u_1 \end{cases} \tag{3}$$

with the initial conditions as:

$$u_1(x,0)=sh(x), u_2(x,0)=ch(x) \tag{4}$$

Let

$$u_1(x,t)=a_{10}(x)+a_{11}(x)t^{0.5}+a_{12}t+\cdots \tag{5}$$

$$u_2(x,t)=a_{20}(x)+a_{21}(x)t^{0.2}+a_{22}t^{0.4}+\cdots \tag{6}$$

If these two functions $u_1(x, t)$ and $u_2(x, t)$ are the solutions of the previous problem, then we have by initial conditions:

$$a_{10}(x)=sh(x), a_{20}(x)=ch(x)$$

Furthermore, substituting eqs. (5) and (6) into eq. (3) yields:

$$sh(x)=ch(x)$$

This is a contradiction. So that the functions eqs. (5) and (6) are not the solution of the eq. (3) system.

Previous analysis lead us to propose the following algorithm for obtaining the solution of the systems of eqs. (1) and (2):

Step 1: Let

$$u_1(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}a_{ij}(x)t^{i\alpha_1+j\alpha_2} \tag{7}$$

$$u_2(x,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{ij}(x)t^{i\alpha_1+j\alpha_2} \tag{8}$$

Step 2: By initial conditions, we obtain:

$$a_{00}(x)=\phi_1(x)$$

$$b_{00}(x)=\phi_2(x)$$

Step 3: Take

$$a_{0k}=b_{k0}=0.\ \ (k=0,1,2,\cdots)$$

Step 4: The remaining coefficients $a_{ij}(x)$ and $b_{ij}(x)$ can be determined by substituting eqs. (7) and (8) into eq. (1), and equating the coefficients of $t^{i\alpha_1+j\alpha_2}$ in both sides.

Thus we can obtain the solution of the systems eqs. (1) and (2) in the forms of eqs. (7) and (8).

Next, let us consider two cases which are often used in applications.

Case 1. Consider the following linear systems:

$$\begin{cases} \mathrm{D}_t^{\alpha_1}u_1(x,t)=c_1u_1+c_2u_2+c_3u_{1x}+c_4u_{2x} \\ \mathrm{D}_t^{\alpha_2}u_2(x,t)=d_1u_1+d_2u_2+d_3u_{1x}+d_4u_{2x} \end{cases} \tag{9}$$

with the initial conditions:

$$u_1(x,0)=\phi_1(x), u_2(x,0)=\phi_2(x) \tag{10}$$

where c_i, d_i (i = 1, 2, 3, 4) are constants.

We will seek the solutions of eqs. (9) and (10) in the forms of eqs. (7) and (8).

According to the previous algorithm, we get:

$$a_{00}(x)=\phi_1(x), \quad b_{00}(x)=\phi_2(x)$$

$$a_{0k}(x)=b_{k0}(x)=0, \quad (k=1,2,\cdots)$$

$$a_{l\,p}(x)=\frac{\Gamma(1+(l-1)\alpha_1+p\alpha_2)}{\Gamma(1+l\alpha_1+p\alpha_2)}(c_1a_{l-1,p}+c_2b_{l-1,p}+c_3a'_{l-1,p}+c_4b'_{l-1,p}), \ (l\ge 1, p\ge 0)$$

$$b_{r\,s}(x)=\frac{\Gamma(1+(s-1)\alpha_2+r\alpha_1)}{\Gamma(1+r\alpha_1+s\alpha_2)}(d_1a_{r,s-1}+d_2b_{r,s-1}+d_3a'_{r,s-1}+d_4b'_{r,s-1}), \ (r\ge 0, s\ge 1)$$

Hence we can obtain the solution of eqs. (9) and (10) by previous iteration formulas.

For example, if we take:

$$c_3=d_4=0, \quad c_1=c_2=d_1=d_2=-1, \quad c_4=d_3=1, \ \text{and} \ \phi_1(x)=sh(x), \quad \phi_2(x)=ch(x)$$

then we have:

$$u_1(x,t)=sh(x)-\frac{ch(x)}{\Gamma(1+\alpha_1)}t^{\alpha_1}+\frac{ch(x)}{\Gamma(1+2\alpha_1)}t^{2\alpha_1}+\frac{sh(x)-ch(x)}{\Gamma(1+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2}+\cdots$$

$$u_2(x,t)=ch(x)-\frac{sh(x)}{\Gamma(1+\alpha_2)}t^{\alpha_2}+\frac{sh(x)}{\Gamma(1+2\alpha_2)}t^{2\alpha_2}+\frac{ch(x)-sh(x)}{\Gamma(1+\alpha_1+\alpha_2)}t^{\alpha_1+\alpha_2}+\cdots$$

When $\alpha_1=\alpha_2=1$ then:

$$u_1(x,t)=sh(x)\left(1+\frac{t^2}{2!}+\cdots\right)-ch(x)\left(t+\frac{t^3}{3!}+\cdots\right)$$

$$u_2(x,t)=ch(x)\left(1+\frac{t^2}{2!}+\cdots\right)-sh(x)\left(t+\frac{t^3}{3!}+\cdots\right)$$

which are exactly solutions of the following problem [13]:

$$\begin{cases}\dfrac{\partial}{\partial t}u_1(x,t)=\dfrac{\partial}{\partial x}u_2-u_1-u_2\\[2mm] \dfrac{\partial}{\partial t}u_2(x,t)=\dfrac{\partial}{\partial x}u_1-u_1-u_2\end{cases}$$

with the initial conditions:

$$u_1(x,0)=sh(x), u_2(x,0)=ch(x)$$

Case 2. Consider the following non-linear systems:

$$\begin{cases} D_t^{\alpha_1} u_1(x,t) = h_1(x) - u_1 - u_{1x}u_2 \\ D_t^{\alpha_2} u_2(x,t) = h_2(x) + u_2 + u_1 u_{2x} \end{cases} \tag{11}$$

with the initial conditions:

$$u_1(x,0) = \phi_1(x), u_2(x,0) = \phi_2(x) \tag{12}$$

Similar to *Case 1*, we will seek the solutions of eqs. (11) and (12) in the forms of eqs. (7) and (8).

Following the previous algorithm, we obtain:

$$a_{00}(x) = \phi_1(x), \quad b_{00}(x) = \phi_2(x)$$

$$a_{0k}(x) = b_{k0}(x) = 0, \quad (k = 1, 2, \cdots)$$

$$a_{lp}(x) = \frac{\Gamma(1+(l-1)\alpha_1 + p\alpha_2)}{\Gamma(1+l\alpha_1 + p\alpha_2)} \left(-a_{l-1,p} - \sum_{i=0}^{l-1} \sum_{j=0}^{p} a'_{ij} b_{l-1-i,p-j} \right), \quad (l > 1, p \geq 0)$$

$$b_{rs}(x) = \frac{\Gamma(1+(s-1)\alpha_2 + r\alpha_1)}{\Gamma(1+r\alpha_1 + s\alpha_2)} \left(b_{r,s-1} - \sum_{i=0}^{r} \sum_{j=0}^{s-1} a_{ij} b'_{r-i,s-1-j} \right), \quad (s > 1, r \geq 0)$$

Thus we can obtain the solution of eqs. (11) and (12) by previous iteration formulas.

For example, if we take:

$$h_1(x) = h_2(x) = 1, \quad \text{and } \phi_1(x) = e^x, \phi_2(x) = e^{-x}$$

then we find that the solutions are:

$$u_1(x,t) = e^x - \frac{e^x t^{\alpha_1}}{\Gamma(1+\alpha_1)} + \frac{(e^x + 1)t^{2\alpha_1}}{\Gamma(1+2\alpha_1)} - \frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \cdots$$

$$u_2(x,t) = e^{-x} + \frac{e^{-x} t^{\alpha_2}}{\Gamma(1+\alpha_2)} + \frac{(e^{-x} - 1)t^{2\alpha_2}}{\Gamma(1+2\alpha_2)} + \frac{t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} + \cdots$$

If $\alpha_1 = \alpha_2 = 1$ we obtain the solutions:

$$u_1(x,t) = e^x \left(1 - t + \frac{t^2}{2!} - \cdots \right) = e^{x-t}$$

$$u_2(x,t) = e^{-x} \left(1 + t + \frac{t^2}{2!} + \cdots \right) = e^{-x+t}$$

which are the exactly solutions of the following problem [13]:

$$\begin{cases} \dfrac{\partial}{\partial t} u_1(x,t) = 1 - u_1 - \dfrac{\partial u_1}{\partial x} u_2 \\ \dfrac{\partial}{\partial t} u_2(x,t) = 1 + u_2 + u_1 \dfrac{\partial u_2}{\partial x} \end{cases}$$

with the initial conditions:

$$u_1(x,0) = \mathrm{e}^x, \qquad u_2(x,0) = \mathrm{e}^{-x}$$

Conclusion

The main aim of this work is to provide a reliable algorithm for the solutions to the systems of fractional differential equations by using the FPSM. This goal is achieved. It is emphasized that the form of solutions must be chosen appropriately.

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