APPLICATION OF LOCAL FRACTIONAL FOURIER SINE TRANSFORM FOR 1-D LOCAL FRACTIONAL HEAT TRANSFER EQUATION

by

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This paper proposes a new method called the local fractional Fourier sine transform to solve fractional differential equations on a fractal space. The method takes full advantages of the Yang-Fourier transform, the local fractional Fourier cosine, and sine transforms. A 1-D local fractional heat transfer equation is used as an example to reveal the merits of the new technology, and the example can be used as a paradigm for other applications.

Key words: *fractal space, local fractional Fourier sine transform, local fractional Fourier cosine transform, heat transfer equation, local fractional derivative*

Introduction

Fractional Fourier transform (FRFT) in the form of fractional powers of the Fourier operator appeared in the mathematical literature between the two world wars [1, 2]. Later on, this notion has been applied particularly in signal processing, chemistry, optics, quantum mechanics, dynamical systems, and stochastic processes. On account of the importance of the FRFT, a lot of its generalizations have been introduced [3-7]. For example, Zayed [4] extended the FRFT to larger classes of functions and generalized functions. Zayed [5] introduced a new class of fractional integral transforms, including the FRFT and the fractional Hankel transforms, *etc.* A detailed survey on the mathematical background, properties, and applications related to the FRFT is provided in [6]. Luchko *et al.* [7] established a new FRFT of the real order α ($0 < \alpha \leq 1$). Recently, Jumarie [8] introduced Fourier transform of fractional order via the Mittag-Leffler function. Very recently, Yang [9, 10] established the Yang-Fourier transform based on local fractional calculus. The Yang-Fourier transform is a powerful and effective mathematical tool for solving fractional differential equations involving local fractional derivative [11].

Local fractional calculus is one of the most powerful tools to study everywhere continuous but nowhere differentiable functions in areas ranging from fundamental science to en-

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gineering [9, 10, 12-25]. For these advantages, local fractional calculus was successfully applied in local fractional Fourier analysis [9, 10], the local fractional Laplace problems [9, 10], local fractional short time transform [9, 10], local fractional wavelet transform [9, 10, 23], fractal signal [23, 24], and local fractional variational calculus [25], local fractional Stieltjes transform [26], local fractional improper integral [27], mean value theorems for local fractional integrals [28], and some local fractional integral inequalities [29]. More details for the local fractional calculus and its applications are available in [9-33].

The main aim of this paper is to apply local fractional sine transforms to solve the heat transfer equation with local fractional derivative.

Preliminaries

In this section, we introduce some mathematical fundamentals of local fractional calculus and recall the basic notions of local fractional continuity, local fractional derivative, and local fractional integral of non-differential functions.

Local fractional continuity of functions

Lemma 1. Assume that F is a subset of the real line and be a fractal. Let $f : (F, d)\to(\Omega', d')$ be a bi-Lipschitz mapping. Then there exist two positive constants ρ, τ, and $F \subset R$, [25]:

$$\rho^s H^s(F) \le H^s\left[f(F)\right] \le \tau^s H^s(F) \tag{1}$$

such that for all $x_1, x_2 \in F$:

$$\rho^\alpha \mid x_1 - x_2 \mid^\alpha \le \mid f(x_1) - f(x_2) \mid \le \tau^\alpha \mid x_1 - x_2 \mid^\alpha \tag{2}$$

From *Lemma 1*, we get:

$$\mid f(x_1) - f(x_2) \mid \le \tau^\alpha \mid x_1 - x_2 \mid^\alpha \tag{3}$$

such that:

$$\mid f(x_1) - f(x_2) \mid \le \varepsilon^\alpha \tag{4}$$

where α is fractal dimension of F. The result that is directly deduced from fractal geometry is related to fractal coarse-grained mass function $\gamma^\alpha[F, a, b]$ which reads [25]:

$$\gamma^\alpha[F,a,b] = \frac{H^\alpha\left[F \cap (a,b)\right]}{\Gamma(1+\alpha)} \tag{5}$$

with

$$H^\alpha\left[F \cap (a,b)\right] = (b-a)^\alpha \tag{6}$$

where H^α is α dimensional Hausdorff measure.

Definition 1. If there exists [10, 25]:

$$\mid f(x) - f(x_0) \mid \le \varepsilon^\alpha \tag{7}$$

with $|x = x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x\to x_0} f(x) = f(x_0)$. The $f(x)$ is called local fractional continuous on the interval (a, b), denoted by:

$$f(x) \in C_\alpha(a,b) \tag{8}$$

if eq. (7) is valid for $x \in (a,b)$.

Definition 2. Assume that function $f(x)$ is a non-differentiable function of exponent α, $0 < \alpha \le 1$, which satisfies Hölder function of exponent α, then for $x, y \in X$ such that [10, 25]:

$$| f(x) - f(y) | \le C | x - y |^\alpha \tag{9}$$

Local fractional derivatives and integrals

Definition 3. Assume that $f(x) \in C_\alpha(a, b)$. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is given by, [10, 25]:

$$f^{(\alpha)}(x) = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \lim_{x \to x_0} \frac{\Delta^\alpha \left[f(x) - f(x_0) \right]}{(x - x_0)^\alpha} \tag{10}$$

where $\Delta^\alpha[f(x) - f(x_0)] \cong \Gamma(1 + \alpha)\Delta[f(x) - f(x_0)]$.

For any $(x) \in (a, b)$, there exists:

$$f^{(\alpha)}(x) = \mathrm{D}_x^{(\alpha)} f(x)$$

denoted by:

$$f(x) \in \mathrm{D}_x^{(\alpha)}(a,b)$$

Local fractional derivative of high order is derived [25]:

$$f^{(k\alpha)}(x) = \overbrace{\mathrm{D}_x^{(\alpha)} \cdots \mathrm{D}_x^{(\alpha)}}^{k \text{ times}} f(x)$$

and local fractional partial derivative of high order is derived [25]:

$$\frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x)$$

Definition 4. Assume that $f(x) \in C_\alpha(a, b)$. Local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is established by [10, 25]:

$${}_a I_b^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \ldots \Delta t_{N-1}\}$, and $[t_j, t_{j+1}]$, $j = 1, \ldots, N - 1$, $t_0 = a$, $t_N = b$ is a partition of the interval $[a, b]$.

Special functions in fractal space

Definition 5. The Mittag-Leffler function in fractal space is given by [10, 25]:

$$E_\alpha(x^\alpha) := \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad x \in R, 0 < \alpha \le 1$$

Definition 6. The sine function in fractal space is established by [10, 25]:

$$\sin_\alpha x^\alpha := \sum_{k=0}^{\infty}(-1)^k \frac{x^{\alpha(2k+1)}}{\Gamma\left[1+\alpha(2k+1)\right]}, \quad x\in R, 0<\alpha\le 1$$

Definition 7. The cosine function in fractal space is given by [10, 25]:

$$\cos_\alpha x^\alpha := \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}, \quad x\in R, \quad 0<\alpha\le 1$$

The following rules hold:

$$E_\alpha(x^\alpha)E_\alpha(y^\alpha)=E_\alpha\left[(x+y)^\alpha\right], \quad E_\alpha(x^\alpha)E_\alpha(-y^\alpha)=E_\alpha\left[(x-y)^\alpha\right]$$

Local fractional cosine and sine transforms

In this section, we begin with the following result [9, 10]:

$$f(x)=\frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} C_k E_\alpha(i^\alpha x^\alpha \omega^\alpha)(\mathrm{d}\omega)^\alpha \tag{11}$$

where

$$C_k=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(x)E_\alpha(-i^\alpha x^\alpha \omega^\alpha)(\mathrm{d}x)^\alpha \tag{12}$$

From eq. (12), the Yang-Fourier transform of $f(x)$ is given by [9, 10]:

$$F_\alpha\{f(x)\}=f_\omega^{F,\alpha}(\omega):=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} E_\alpha(-i^\alpha \omega^\alpha x^\alpha)f(x)(\mathrm{d}x)^\alpha \tag{13}$$

and its inverse formula of Yang-Fourier's transforms is defined:

$$f(x)=F_\alpha^{-1}\left[f_\omega^{F,\alpha}(\omega)\right]:=\frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} E_\alpha(i^\alpha \omega^\alpha x^\alpha)f_\omega^{F,\alpha}(\omega)(\mathrm{d}\omega)^\alpha \tag{14}$$

Now, by eqs. (11) and (12), we have:

$$f(x)=\frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty}\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(\xi)E_\alpha(-i^\alpha \xi^\alpha \omega^\alpha)(\mathrm{d}\xi)^\alpha\right]E_\alpha(i^\alpha x^\alpha \omega^\alpha)(\mathrm{d}\omega)^\alpha \tag{15}$$

Here, we named eq. (15) the Yang-Fourier integral formula.

Based on eqs. (11)-(15), Chen [34] established local fractional Fourier cosine and sine transforms:

$$F_{\alpha,c}\{f(x)\}=f_{\omega,c}^{F,\alpha}(\omega):=\frac{2}{\Gamma(1+\alpha)}\int_{0}^{\infty} f(x)\cos_\alpha(\omega^\alpha x^\alpha)(\mathrm{d}x)^\alpha \tag{16}$$

$$F_{\alpha,s}\{f(x)\}=f_{\omega,s}^{F,\alpha}(\omega):=\frac{2}{\Gamma(1+\alpha)}\int_{0}^{\infty} f(x)\sin_\alpha(\omega^\alpha x^\alpha)(\mathrm{d}x)^\alpha \tag{17}$$

$$f(x)=F_{\alpha.c}^{-1}\left[f_{\omega,c}^{F,\alpha}(\omega)\right]:=\frac{2}{(2\pi)^{\alpha}}\int_{0}^{\infty}f_{\omega,c}^{F,\alpha}(\omega)\cos_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}\omega)^{\alpha} \tag{18}$$

$$f(x)=F_{\alpha.s}^{-1}\left[f_{\omega,s}^{F,\alpha}(\omega)\right]:=\frac{2}{(2\pi)^{\alpha}}\int_{0}^{\infty}f_{\omega,s}^{F,\alpha}(\omega)\sin_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}\omega)^{\alpha} \tag{19}$$

Next, we recall some properties of local fractional Fourier cosine and sine transforms.

Theorem 1. Assume that $F_{\alpha,c}\{f(x)\}=f_{\omega,c}^{F,\alpha}(\omega)$ and $F_{\alpha,s}\left[f(x)\right]=f_{\omega,s}^{F,\alpha}(\omega)$, then, [34]:

$$F_{\alpha,c}\left[f(ax)\right]=\frac{1}{a^{\alpha}}f_{\omega,c}^{F,\alpha}\left(\frac{\omega}{a}\right),\quad F_{\alpha,s}\left[f(ax)\right]=\frac{1}{a^{\alpha}}f_{\omega,s}^{F,\alpha}\left(\frac{\omega}{a}\right)$$

Theorem 2. Convolution *Theorem* for the local fractional Fourier cosine transform, [34].

Assume that $F_{\alpha,c}\left[f(x)\right]=f_{\omega,c}^{F,\alpha}(\omega)$ and $F_{\alpha,c}\left[g(x)\right]=g_{\omega,c}^{F,\alpha}(\omega)$, then:

$$\frac{2}{(2\pi)^{\alpha}}\int_{0}^{\infty}f_{\omega,c}^{F,\alpha}(\omega)g_{\omega,c}^{F,\alpha}(\omega)\cos_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}\omega)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\infty}f(\xi)[g(x+\xi)+g(|x-\xi|)](\mathrm{d}\xi)^{\alpha}$$

Similarly, we obtain:

$$\int_{0}^{\infty}f_{\omega,s}^{F,\alpha}(\omega)g_{\omega,s}^{F,\alpha}(\omega)\cos_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}\omega)^{\alpha}=\frac{(2\pi)^{\alpha}}{2\Gamma(1+\alpha)}\int_{0}^{\infty}f(\xi)[g(\xi+x)+g(\xi-x)](\mathrm{d}\xi)^{\alpha}$$

Applications

In this section, we use the local fractional Fourier sine transform to solve the following 1-D heat transfer equation:

$$\begin{cases}\dfrac{\partial^{\alpha}u}{\partial t^{\alpha}}=a^{2}\dfrac{\partial^{2\alpha}u}{\partial x^{2\alpha}}(0<x<+\infty,\quad t>0)\\ u|_{x=0}=\varphi(t)\\ u|_{t=0}=0\end{cases} \tag{20}$$

Making use of local fractional Fourier sine transform eq. (17), we have:

$$F_{\alpha,s}\left[u(x,t)\right]=\frac{2}{\Gamma(1+\alpha)}\int_{0}^{\infty}u(x,t)\sin_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}x)^{\alpha}=U_{\omega,s}^{F,\alpha}(\omega,t)$$

$$F_{\alpha,s}\left[\frac{\partial^{2}u(x,t)}{\partial x^{2}}\right]=\frac{2}{\Gamma(1+\alpha)}\int_{0}^{\infty}u(x,t)\sin_{\alpha}(\omega^{\alpha}x^{\alpha})(\mathrm{d}x)^{\alpha}=\omega^{\alpha}u|_{x=0}-\omega^{2\alpha}U_{\omega,s}^{F,\alpha}(\omega,t)$$

$$F_{\alpha,s}\{u|_{t=0}\}=U_{\omega,s}^{F,\alpha}(\omega,t)|_{t=0},\quad F_{\alpha,s}\left[\frac{\partial u(x,t)}{\partial t}\right]=\frac{\mathrm{d}^{\alpha}U_{\omega,s}^{F,\alpha}(\omega,t)}{\mathrm{d}t^{\alpha}}$$

From the previous discussion, eq. (20) can be transformed into the following initial problem with parameter ω^{α}:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha U_{\omega,s}^{F,\alpha}(\omega,t)}{\mathrm{d}t^\alpha} = a^2[\omega^\alpha \varphi(t) - \omega^{2\alpha} U_{\omega,s}^{F,\alpha}(\omega,t)] \\ U_{\omega,s}^{F,\alpha}(\omega,t)|_{t=0} = 0 \end{cases} \tag{21}$$

Then, the solution to eq. (21) is:

$$U_{\omega,s}^{F,\alpha}(\omega,t) = E_\alpha(-a^2\omega^{2\alpha}t^\alpha)\frac{1}{\Gamma(1+\alpha)}\int_0^t a^2\omega^\alpha\varphi(s)E_\alpha(a^2\omega^{2\alpha}s^\alpha)\mathrm{d}s$$

By local fractional inverse Fourier sine transform eq. (19), we obtain the solution to the heat transfer equation.

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