SOLVING TIME-SPACE FRACTIONAL FITZHUGH-NAGUMO EQUATION BY USING HE-LAPLACE DECOMPOSITION METHOD

by

Chunfu WEI*

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan, China

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This paper proposes a new method to solve fractional differential equations, which takes full advantages of He's homotopy perturbation, Laplace transform, and He's polynomials and it is named as He-Laplace decomposition method. The time-space fractional Fitzhugh-Nagumo equation is used as example to elucidate the solution process, and the obtained results are of high accuracy. The new method sheds a new light on analytical approach to fractional calculus.

Key words: *time-space fractional Fitzhugh-Nagumo equation, He-Laplace decomposition method, Caputo derivative*

Introduction

Non-linear phenomena occur in a wide range of apparently different contexts in chemical, physical, biological, and economical systems, and fractional PDE [1-7] are widely used in thermal science to untangle the non-local property of heat conduction in discontinuous medium, and now fractal calculus becomes a hot topic in both thermal science and mathematics [8-13].

In this paper, we study the following time-space fractional Fitzhugh-Nagumo equation:

$$\mathrm{D}_t^{\alpha} u(x,t) = \mathrm{D}_x^{\beta} u(x,t) - \mu u + (1+\mu)u^2 - u^3, \quad 0 < \alpha \le 1, \quad 1 < \beta \le 2 \tag{1}$$

subject to the initial condition:

$$u(x,0) = \phi(x) \tag{2}$$

where D^{ν} denotes the Caputo fractional derivative of order ν, μ – an arbitrary constant, and $\phi(x)$ – a given function.

Equation (1) is an important non-linear PDE for describing transmission of thermal energy in thermodynamics, biology and circuit theory [14-20]. Rida *et al*. [18] obtained some approximate solutions of eq. (1) by using the generalized differential transform method. When $\alpha = 1$, Abbasbandy [15] found exact solutions by using the homotopy analysis method. In the present work, we introduce a comprehensive and more efficient approach by using He-Laplace decomposition method to solve eq. (1). Here the non-linear term is replaced by

* Author's, e-mail: mathwcf@163.com

He polynomials and then the dependent variable components are replaced by their corresponding Laplace transform component of the same index. This technique benefits the properties of the He's polynomials and the efficient algorithm to generate the polynomials quickly [20].

Fractional calculus and Laplace transform

We recall the following basic definitions and properties of fractional calculus theory and Laplace transform which shall be used in this paper.

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_λ, $\lambda \in R$ if there exists a real number $p > \lambda$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C\,[0, \infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\lambda$, $n \in N$.

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x) \in C_\lambda$, $\lambda \geq -1$ is defined:

$$\mathrm{J}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s)\mathrm{d}s$$

$$\mathrm{J}^0 f(x) = f(x)$$

Properties of the operator J^α can be found in [21] and we mention only the following: for $\alpha, \beta \geq 0$, $x > 0$ and $\lambda > -1$:

$$\mathrm{J}^\alpha \mathrm{J}^\beta f(x) = \mathrm{J}^{\alpha+\beta} f(x)$$

$$\mathrm{J}^\alpha \mathrm{J}^\beta f(x) = \mathrm{J}^\beta \mathrm{J}^\alpha f(x)$$

$$\mathrm{J}^\alpha (x^\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(1+\alpha+\lambda)} x^{\lambda+\alpha}$$

Definition 3. The fractional derivative of $f(x)$ in Caputo sense is defined:

$$\mathrm{D}^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s)\mathrm{d}s$$

for $m - 1 < \alpha \leq m$, $m \in N^+$, $x > 0$ and $f \in C_{-1}^m$.

We recall the following basic properties :

$$\mathrm{D}^\alpha \mathrm{J}^\alpha f(x) = f(x)$$

$$\mathrm{J}^\alpha \mathrm{D}^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0$$

Definition 4. The Laplace transform of an almost piecewise continuous function $f(t)$ in $[0, \infty)$ is defined:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t \tag{3}$$

where s is real or complex number.

The following properties can be found in [21]:

$$\mathcal{L}(t^{\gamma}) = \int_0^{\infty} \mathrm{e}^{-st} t^{\gamma} \mathrm{d}t = \frac{\Gamma(1+\gamma)}{s^{\gamma+1}}, \qquad \mathrm{Re}(s) \geq 0, \qquad \mathrm{Re}(\gamma) > 0 \tag{4}$$

$$\mathcal{L}[\mathrm{D}^{\alpha} f(t)] = s^{\alpha} F(s) - s^{\alpha-1} f(0), \qquad 0 < \alpha \leq 1 \tag{5}$$

Analysis of He-Laplace decomposition method

To illustrate the basic idea of this method, we rewrite eq. (1):

$$\mathrm{D}_t^{\alpha} u(x,t) = \mathrm{D}_x^{\beta} u - \mu u + N(u) \tag{6}$$

where

$$N(u) = (1+\mu)u^2 - u^3$$

Taking the Laplace transform with respect to t on both side of eq. (6), we get:

$$\mathcal{L}(\mathrm{D}_t^{\alpha} u) = \mathrm{D}_x^{\beta} \mathcal{L}(u) - \mu \mathcal{L}(u) + \mathcal{L}[N(u)]$$

Using the property (5) of the Laplace transform, we have:

$$\mathcal{L}(u) = \frac{1}{s} u(x,0) + \frac{1}{s^{\alpha}} \left\{ \mathrm{D}_x^{\beta} \mathcal{L}(u) - \mu \mathcal{L}(u) + \mathcal{L}\left[N(u)\right] \right\} \tag{7}$$

Operating with the Laplace inverse an both side of eq. (7) gives:

$$u = u(x,0) + \mathcal{L}^{-1} \left(\frac{1}{s^{\alpha}} \left\{ \mathrm{D}_x^{\beta} \mathcal{L}(u) - \mu \mathcal{L}(u) + \mathcal{L}[N(u)] \right\} \right) \tag{8}$$

Now we let:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{9}$$

and the non-linear term $N(u)$ can be decomposed:

$$N\left(\sum_{i=0}^{\infty} u_i \right) = \sum_{i=1}^{\infty} H_i(u) \tag{10}$$

for some He's polynomials $H_n(u)$ [20] that are given by:

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{k=0}^{n} p^k u_k \right) \right]_{p=0}, \qquad n = 0,1,2,\cdots$$

Substituting eqs. (9) and (10) into eq. (8), we get:

$$\sum_{i=0}^{\infty} u_i = u(x,0) + \mathcal{L}^{-1} \left\{ \frac{1}{s^{\alpha}} \left[\mathrm{D}_x^{\beta} \mathcal{L}\left(\sum_{i=0}^{\infty} u_i \right) - \mu \sum_{i=0}^{\infty} \mathcal{L}(u_i) + \sum_{i=0}^{\infty} \mathcal{L}(H_i) \right] \right\} \tag{11}$$

Thus, the components $u_i(x, t)$ of the solution $u(x, t)$ can be computed by using the recursive relation:

$$u_0(x,t) = u(x,0)$$

$$u_1(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathrm{D}_x^\beta \mathcal{L}(u_0) - \mu\mathcal{L}(u_0) + \mathcal{L}(H_0)\right]\right\}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}\left[\mathrm{D}_x^\beta \mathcal{L}(u_1) - \mu\mathcal{L}(u_1) + \mathcal{L}(H_1)\right]\right\} \tag{12}$$

and so on.

Then k-term approximate solution of eq. (6) is given by:

$$u = u_0 + u_1 + \cdots + u_{k-1} \tag{13}$$

Next, as an example, let us consider a typical case $\beta = 2$ in eq. (6):

$$\mathrm{D}_t^\alpha u(x,t) = \frac{\partial^2 u}{\partial x^2} - \mu u + (1+\mu)u^2 - u^3 \qquad 0 < \alpha \le 1$$

subject to the initial condition:

$$u(x,0) = \frac{1}{1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}}$$

Based on the algorithm (12), we get:

$$u_0(x,t) = \frac{1}{1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}}$$

$$u_1(x,t) = \frac{1-2\mu}{2}\frac{\mathrm{e}^{-\frac{x}{\sqrt{2}}}}{\left(1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}\right)^2}\frac{t^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t) = \left(\frac{1-2\mu}{2}\right)^2 \frac{\mathrm{e}^{-\frac{x}{\sqrt{2}}}\left(3+\mathrm{e}^{-\frac{x}{\sqrt{2}}}\right)}{\left(1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}\right)^3}\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}$$

and so on. Thus we obtain the series form solutions:

$$u(x,t) = \frac{1}{1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}} + \frac{1-2\mu}{2}\frac{\mathrm{e}^{-\frac{x}{\sqrt{2}}}}{\left(1+\mathrm{e}^{-\frac{x}{\sqrt{2}}}\right)^2}\frac{t^\alpha}{\Gamma(1+\alpha)} + \cdots$$

which is agreement with the result obtained in [18].

Conclusion

In this work, the He-Laplace decomposition method has been successfully applied to obtain the approximate solution of the time-space fractional Fitzhugh-Nagumo equations. The results show that the proposed algorithm is very efficient, simple and can be applied to other fractional non-linear differential equations.

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