MARKOV CHAIN MONTE CARLO METHOD TO SOLVE FREDHOLM INTEGRAL EQUATIONS

by

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A thermal problem can be always modeled using an integral equation. This paper uses the Monte Carlo method based on the simulation of a continuous Markov chain to solve Fredholm integral equations of the second kind. Some examples are given to show the efficiency of the present work.

Key words: *Monte Carlo method, Fredholm integral equation, Markov chain*

Introduction

The Monte Carlo method is preferable for solving both high-dimensional multiple integrals and large sparse systems of linear algebraic equations [1-3]. Recently, Farnoosh and Ebrahimi [4] employed Monte Carlo method based on the simulation of a continuous Markov chain for solving Fredholm integral equations of the second kind. In this paper, we will consider the following Fredholm integral equation of the second kind:

$$u(x) = f(x) + \lambda \int_0^1 k(x,t)u(t)\mathrm{d}t, \qquad 0 \le x \le 1 \tag{1}$$

where the function $f(x) \in L_2[0,1]$, the kernel $k(x, t) \in L_2([0, 1] \times [0, 1])$ is given and $u(x) \in L_2[0,1]$ is the unknown to determined.

In eq. (1), $u(x)$ can be considered as temperature distribution along a fin. This paper uses a continuous Markov chain with state space [0, 1] for simulation.

Overview of the method

Equation (1) may be written in the operational form:

$$u(x) = f(x) + (Ku)(x), \tag{2}$$

where

$$(Ku)(x) = \int_0^1 K(x,t)u(t)\mathrm{d}t, \tag{3}$$

proceeding recursively, we obtain:

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$$K[(Ku)](x) = (K^2u)(x) = \int_0^1\int_0^1 k(x,t)k(t,t_1)u(t_1)\mathrm{d}t\mathrm{d}t_1 \tag{4}$$

$$\vdots$$

$$K[(K^{n-1}u)](x) = (K^n u)(x) = \int_0^1 k(x,t_{n-1})K^{n-1}u(t_{n-1})\mathrm{d}t_{n-1} \tag{5}$$

Let us assume that:

$$|K| = \underbrace{\sup}_{[0,1]}\int_0^1 |k(x,t)|\mathrm{d}t < 1 \tag{6}$$

under this assumption, we can solve eq. (1) by applying the following recursive equation:

$$u^{(n+1)}(x) = (Ku^n)(x) + f(x), \qquad n = 1, 2, \cdots \tag{7}$$

if $u^{(0)}(x) = 0$ and $K^0 \equiv 0$ then from eq. (7), we obtain:

$$u^{(n+1)}(x) = f(x) + (Kf)(x) + \cdots + (K^n f)(x) = \sum_{m=0}^{n}(K^m f)(x) \tag{8}$$

It is well known that:

$$\lim_{n\to\infty} u^{(n)}(x) = \lim_{n\to\infty}\sum_{m=0}^{n}(K^m f)(x) = u(x) \tag{9}$$

Continuous Markov chain Monte Carlo method

Consider the continuous Markov chain:

$$P = \| P(x,y) \| \tag{10}$$

with state space [0,1], satisfying:

$$\int_0^1 P(x,y)\mathrm{d}y = 1 \tag{11}$$

and

$$\int_0^1 p(x)\mathrm{d}x = 1 \tag{12}$$

where $p(x)$ and $P(x, y)$ are the initial and the transition densities of the Markov chain (10)-(12), respectively.

Define the weight function, W_m, for Markov chain, (10)-(12) using recursion formula:

$$W_m = W_{m-1}\frac{k(x_{m-1}, x_m)}{P(x_{m-1}, x_m)}, \qquad m = 1, 2, \cdots \tag{13}$$

where $W_0 = 1$.

Define the following random variable:

$$\Gamma_n[h] = \sum_{m=0}^{n} W_m f(x_m) \tag{14}$$

associated with the sample path:

$$x_0 \to x_1 \to x_2 \to \cdots \to x_n \tag{15}$$

where n is a given integer number.

Definition 1. The inner product of two function $h(x)$ and $u(x)$ is defined by:

$$\langle h, u \rangle = \int_0^1 h(x)u(x)\mathrm{d}x \tag{16}$$

Theorem 1. The mathematical expectation value of random variable $\Gamma_n[h]$ is equal to the inner product $\langle h, u^{(n+1)} \rangle$, *i. e.*:

$$E(\Gamma_n[h]) = \langle h, u^{(n+1)} \rangle \tag{17}$$

Proof 1. Each path $x_0 \to x_1 \to x_2 \to \ldots x_n$, will be realized with probability:

$$P(x_0, x_1, \cdots, x_n) = P(x_0, x_1)P(x_1, x_2)\cdots P(x_{n-1}, x_n) \tag{18}$$

While simulating the Markov chain (10)-(12), since the random variable $\Gamma_n[h]$ is defined along path $x_0 \to x_1 \to x_2 \to \ldots x_n$, we have:

$$E(\Gamma_n[h]) = \int_0^1 \int_0^1 \cdots \int_0^1 \Gamma_n[h] P(x_0, x_1)P(x_1, x_2)\cdots P(x_{n-1}, x_n)\mathrm{d}x_0\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{19}$$

which, together with eqs. (13) and (14), gives:

$$E(\Gamma_n[h]) = E\left[\sum_{m=0}^{n} W_m f(x_m)\right] = \int_0^1 \int_0^1 \cdots \int_0^1 \sum_{m=0}^{n} \Phi\Psi f(x_m)p(x_m)\mathrm{d}x_0\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{20}$$

where

$$\Phi = k(x_0, x_1)k(x_1, x_2)\cdots k(x_{m-1}, x_m) \tag{21}$$

and

$$\Psi = P(x_m, x_{m+1})\cdots P(x_{n-1}, x_n) \tag{22}$$

Using the property $\int_0^1 P(x, y)\mathrm{d}y = 1$, the eq. (20) can be written:

$$E(\Gamma_n[h]) = \sum_{m=0}^{n} \int_0^1 \int_0^1 \cdots \int_0^1 \sum_{m=0}^{n} \Phi f(x_m)\mathrm{d}x_0\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{23}$$

thus, taking into account eqs. (5), (8), and (16), we obtain:

$$E(\Gamma_n[h]) = \left\langle h(x), (K^m f)(x) \right\rangle = \left\langle h(x), u^{(n+1)}(x) \right\rangle \tag{24}$$

Computational procedure for estimating $\langle h, u^{(n+1)} \rangle$

To estimate $\langle h, u^{(n+1)} \rangle$,we consider a continuous Markov chain with transition kernel:

$$P(x, y) = \rho(x)\delta(y - x) + [1 - \rho(x)]g(y), \qquad x, y \in [0,1] \tag{25}$$

where $\delta(y - x)$ is the Dirac's delta function at x, $g(x)$ – a probability density on [0,1] and $\rho(x)$ – a function such that $0 < \rho(x) < 1$ and:

$$\int_0^1 \frac{g(x)}{1 - \rho(x)} \mathrm{d}x < \infty$$

Step 1. Choose any integer n and then we simulate N independent random paths of length n:

$$x_0^{(s)} \to x_1^{(s)} \to x_2^{(s)} \to \cdots \to x_n^{(s)}, \qquad s = 1, 2, \cdots, N \tag{26}$$

from the Markov chain (25).

Algorithm 1

1. *Input initial data:* the number of random paths N, the length of Markov chain n, the function $\rho(x)$,the probability density g on [0,1].

2. For s: = 1 to N do step 2.1.

2.1. *Perform one random path:*

2.1.1. Give $x_0 \in [0, 1]$

2.1.2. For m: = 0 to $n - 1$ do

2.1.2.1. Generate an uniformly distributed random number $\tau \in (0, 1)$;

2.1.2.2. If $\rho(x_m^{(s)}) > \tau$ then, calculate $x_{m+1}^{(s)} = x_m^{(s)}$;

else, generate $x_{m+1}^{(s)}$ from probability density g

2.1.3. *End of the random path.*

3. *Output* $u(x_0)$.

4. *End of the Algorithm 1.*

Step 2. We also define the random variable $\Gamma_n^{(s)}[h]$ on the path (26) such that:

$$\Gamma_n^{(s)}[h] = \sum_{m=0}^{n} W_m^{(s)} f(x_m) \tag{27}$$

where

$$W_m^{(s)} = \frac{k\left(x_0^{(s)}, x_1^{(s)}\right) k\left(x_1^{(s)}, x_2^{(s)}\right) \cdots k\left(x_{n-1}^{(s)}, x_n^{(s)}\right)}{P\left(x_0^{(s)}, x_1^{(s)}\right) P\left(x_1^{(s)}, x_2^{(s)}\right) \cdots P\left(x_{n-1}^{(s)}, x_n^{(s)}\right)}, \qquad m = 1, 2, \cdots, \tag{28}$$

and $W_0^{(s)} = 1$.

Step 3. Finally evaluate the sample mean:

$$\Theta_n[h] = \frac{1}{n}\sum_{s=1}^{N} \Gamma_n^{(s)}[h] \approx \left\langle h, u^{(n+1)} \right\rangle \tag{29}$$

Discussion of the numerical experiments

To give a clear overview of the present Monte Carlo (MC) method, the following examples will be considered and the solution of which is to be obtained.

Example 1. Consider the following equation:

$$u(x) = \sqrt{x} - \int_0^1 \sqrt{xt}\, u(t)\mathrm{d}t \tag{30}$$

for which the exact solution is $u(x) = 2/3\, x^{1/2}$.

The result obtained for $u(x)$ with $N = 1000$, $\alpha = 2.02$, and $\beta = 1$ are presented in tab. 1 and fig. 1.

Table 1. *N* = 1000, *n* = 60

x	MC	Exact
0.1	0.2107	0.2108
0.2	0.2981	0.2981
0.3	0.3650	0.3651
0.4	0.4216	0.4216
0.5	0.4712	0.4714
0.6	0.5163	0.5164
0.7	0.5578	0.5578
0.8	0.5963	0.5963
0.9	0.6325	0.6325
1.0	0.6665	0.6667

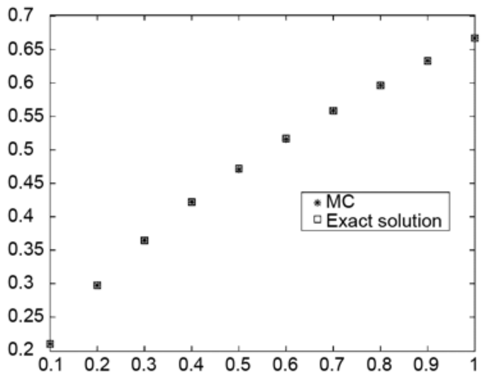

Figure 1. The solution of *Example 1* with MC in case *N* = 1000, *n* = 60

Example 2. Consider the following equation:

$$u(x) = 0.9x^2 + \int_0^1 0.5x^2t^2u(t)\mathrm{d}t \tag{31}$$

for which the exact solution is $u(x) = x^2$.

The result obtained for $u(x)$ with $N = 100$, $\alpha = 2.02$, and $\beta = 1$ are presented in tab. 2 and fig. 2.

Example 3. Consider the following equation:

$$u(x) = \frac{5x}{6} + \frac{1}{2}\int_0^1 xt\, u(t)\mathrm{d}t \tag{31}$$

for which the exact solution is $u(x) = x$.

The result obtained for $u(x)$ with $N = 1000$, $\alpha = 2.05$, and $\beta = 1$ are presented in tab. 3 and fig. 3.

Table 2. ***N*** **= 100,** ***n*** **= 55**

x	MC	Exact
0.1	0.01009	0.01
0.2	0.03979	0.04
0.3	0.08949	0.09
0.4	0.15913	0.16
0.5	0.25177	0.25
0.6	0.35914	0.36
0.7	0.48278	0.49
0.8	0.65513	0.64
0.9	0.81568	0.81
1.0	1.01747	1.00

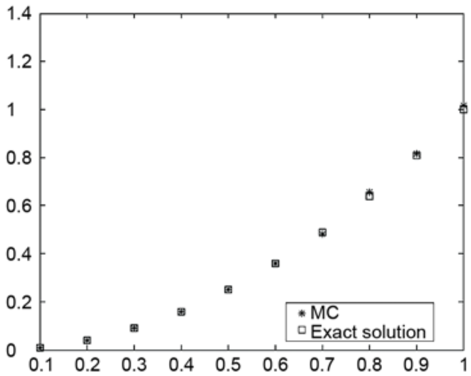

Figure 2. The solution of *Example 2* with MC in case *N* = 100, *n* = 55

Table 3. ***N*** **= 1000,** ***n*** **= 30**

x	MC	Exact
0.1	0.0990	0.1
0.2	0.1985	0.2
0.3	0.2974	0.3
0.4	0.3960	0.4
0.5	0.4902	0.5
0.6	0.5953	0.6
0.7	0.7097	0.7
0.8	0.7996	0.8
0.9	0.8884	0.9
1.0	0.9484	1.0

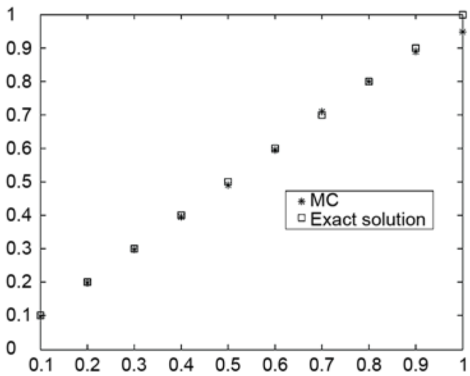

Figure 3. The solution of *Example 3* with MC in case *N* = 1000, *n* = 30

Conclusion

The present study, successfully applied the continuous Markov chain Monte Carlo method for the solution of Fredholm integral equations of the second kind. From the numerical examples it can be seen that the proposed Monte Carlo method is efficient and accurate to estimate the solution of Fredholm integral eqs. (1).

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