EXACT SOLUTIONS WITH EXTERNAL LINEAR FUNCTIONS FOR THE POTENTIAL YU-TODA-SASA-FUKUYAMA EQUATION

by

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Constructing exact solutions of non-linear PDE is of both theoretical and practical values. In this paper, a modified F-expansion method is proposed to construct exact solutions of non-linear PDE. To illustrate the validity and advantages of the proposed method, a (3+1)-D potential Yu-Toda-Sasa-Fukuyama equation is considered and more general exact solutions with external linear functions are obtained including Jacobi elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions. It is shown that the original F-expansion method can not construct exact solutions of the potential Yu-Toda-Sasa-Fukuyama equation but the modified method is valid. The modified F-expansion method can lead to such exact solutions with external linear functions which possess a remarkable dynamical property, which is the solitary wave does not propagate in the horizontal direction as the traditional waves. The modified F-expansion method can be used for exactly solving some other non-linear PDE.

Key words: *potential Yu-Toda-Sasa-Fukuyama equation, modified F-expansion method, Jacobi elliptic function solution, hyperbolic function solution, trigonometric function solution*

Introduction

As we know that non-linear PDE are often used to describe some non-linear phenomena of the real world involved in many fields from physics to biology, economics, chemistry, mechanics, fluid dynamics, engineering, control theory, *etc.* Usually, researchers investigate solutions of such non-linear PDE to gain more insight into these physical phenomena for further applications. In the past several decades, searching for exact solutions of non-linear PDE has become one of the most important and significant tasks. Many effective methods for constructing exact solutions of non-linear PDE have been proposed, such as the inverse scattering method [1], Backlund transformation [2], Darboux transformation [3], Hirota's bilinear method [4], Painleve expansion [5], homogeneous balance method [6], auxiliary equation method [7], and exp-function method [8-10].

In 2003, the so-called F-expansion method was proposed by Liu *et al.* [11] to construct exact solutions of non-linear PDE, which can be thought of as an over-all generalization of Jacobi elliptic function expansion method [12-14]. The F-expansion method has been applied to many non-linear PDE like those in [15-17] and was improved in different manners [18-21]. The present paper is motivated by the desire to propose a modified F-expansion method to further extend the application scope of the F-expansion method [11]. In order to il-

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lustrate the validity and advantages of the modified method, in this paper we shall consider the (3+1)-D potential Yu-Toda-Sasa-Fukuyama (pYTSF) equation [22]:

$$-u_{xt} + u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} = 0 \tag{1}$$

It should be noted that eq. (1) can be derived from the (3+1)-D YTSF equation:

$$[-v_t + \Phi(v)v_z]_x + 3v_{yy} = 0, \qquad \Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1} \tag{2}$$

by setting the potential $v = u_x$. The (3+1)-D YTSF eq. (2) was derived by Yu *et al.* [23] in the process of extending the (2+1)-D Bogoyavlenskii-Schif (BS) equation [24]:

$$v_t + \Phi(v)v_z = 0, \qquad \Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1} \tag{3}$$

to (3+1)-D non-linear evolution equation. Thus, there are two ways through which the (1+1)-D Korteweg-de Vries equation:

$$v_t + \Phi(v)v_x = 0, \qquad \Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}$$

can be extended to the (3+1)-D YTSF eq. (2). The first way is through the BS eq. (3), and the other one is through the (2+1)-D Kadomtsev-Petviashvili equation:

$$[-v_t + \Phi(v)v_x]_x + 3v_{yy} = 0, \qquad \Phi(v) = \partial_x^2 + 4v + 2v_x\partial_x^{-1}$$

Modified F-expansion method

In this section, let us outline the basic idea of the modified F-expansion method. For a given non-linear PDE, say, in four variables *x*, *y*, *z*, and *t*:

$$P(x, y, z, t, u, u_x, u_y, u_z, u_t, \cdots) = 0 \tag{4}$$

we first use the following transformation:

$$u = u(\xi), \qquad \xi = ax + by + cz - \omega t \tag{5}$$

where *a*, *b*, *c*, and ω are constants, then eq. (5) is reduced into an ODE:

$$E(x, y, z, t, u, u^{(r)}, u^{(r+1)}, \cdots) = 0 \tag{6}$$

here $u^{(i)} = \mathrm{d}^i u/\mathrm{d}\xi^i$ ($i \geq r$), and *r* is the least order of derivatives in the equation. As pointed out in [25], if *E* is a total ξ-derivative of another function, integrating eq. (6) with respect to ξ we further reduce the transformed equation.

Secondly, we search for exact solutions determined by:

$$u^{(r)}(\xi) = v(\xi) = \sum_{i=1}^{m} \alpha_i F^i(\xi) + \alpha_0, \qquad \alpha_m \neq 0 \tag{7}$$

where α_i (i = 0, 1, 2, …, m) are all constants to be determined, $F(\xi)$ satisfies the auxiliary ODE:

$$F'^2(\xi) = PF^4(\xi)QF^2(\xi) + R \tag{8}$$

and hence holds:

$$\begin{cases} F''(\xi) = 2PF^3(\xi) + QF(\xi) \\ F^{(3)}(\xi) = \left[6PF^2(\xi) + Q\right]F'(\xi) \\ F^{(4)}(\xi) = 24P^2F^5(\xi) + 20PQF^3(\xi) + (Q^2 + 12PR)F(\xi) \\ \qquad \cdots \end{cases} \tag{9}$$

where P, Q, and R are parameters, a direct computation from eq. (7) gives:

$$u^{(r+1)}(\xi) = v'(\xi) = \sum_{i=1}^{m}[i\alpha_i F^{i-1}(\xi)F'(\xi)] \tag{10}$$

$$u^{(r+2)}(\xi) = v''(\xi) = \sum_{i=1}^{m} i(i-1)\alpha_i F^{i-2}F'^2(\xi) + \sum_{i=1}^{m} i\alpha_i F^{i-1}(\xi)F''(\xi) =$$

$$= \sum_{i=1}^{m} i(i-1)\alpha_i F^{i-2}(\xi)[PF^4(\xi) + QF^2(\xi) + R] + \sum_{i=1}^{m} i\alpha_i F^{i-1}(\xi)[2PF^3(\xi) + QF(\xi)] \tag{11}$$

and so on. Here the prime denotes the derivative with respect to ξ.

To determine u explicitly, we take the following four steps.

Step 1. Determine the integer m by substituting eq. (7) along with eq. (8) into eq. (6), and balancing the highest order non-linear term(s) and the highest order partial derivative.

Step 2. Substitute eq. (7) given the value of m determined in *Step 1* along with eqs. (8) and (9) into eq. (6) and collect all terms with the same order of $F^j(\xi)F'^s(\xi)$ (j = 0, 1, 2, …; s = 0, 1) together, the LHS of eq. (6) is converted into a polynomial in $F^j(\xi)F'^s(\xi)$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for a, b, c, ω, α_i (i = 0, 1, 2,…, m).

Step 3. Solve the system of algebraic equations obtained in *Step 2* for a, b, c, ω, α_i (i = 0, 1, 2,…, m) by use of MATHEMATICA.

Step 4. Use the results obtained in previous steps to derive a series of fundamental solutions $v(\xi)$ of eq. (7) depending on $F(\xi)$, since eq. (8) has different types of Jacobi elliptic function solutions [21], then we can obtain exact solutions of eq. (4) by integrating each of the obtained fundamental solutions $v(\xi)$ with respect to ξ, r times:

$$u = u(\xi) = \int^{\xi}\int^{\xi_r}\cdots\int^{\xi_2} v(\xi_1)\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{r-1}\mathrm{d}\xi_r + \sum_{j=1}^{r} d_j\xi^{r-j} \tag{12}$$

where d_j (j = 1, 2,…, r) are arbitrary constants.

Remark 1. It can be easily found that when $r = 0$, $u(\xi) = v(\xi)$, and then eq. (7) belongs to the case used in [11-16]. Under this circumstance, the method proposed in present paper is same as that of [11-16]. However, when $r \geq 1$, eq. (12) maybe contains an explicit polynomial in ξ even if it is simplified. Such a solution can not be obtained by either the F-expansion method [11] or its improvements [17-21], see the example in the next section for more details. Therefore, the proposed method can be seen as a modified version of Liu *et al.*'s [11] F-expansion method.

Exact solutions

Let us consider in this section the (3+1)-D pYTSF eq. (1). We have the following *Theorem 1*.

Theorem 1. Let eq. (8) as the auxiliary ODE, the (3+1)-D pYTSF eq. (1) has a kind of formal solution:

$$u(\xi) = -2aP\int F^2(\xi)\mathrm{d}\xi - \frac{2}{3}a\left(Q \pm \sqrt{Q^2 - 3PR}\right)\xi + c_0 \tag{13}$$

where c_0 is an arbitrary constant, and ξ is determined by:

$$\xi = ax + by + \frac{\left(Q \pm \sqrt{Q^2 - 3PR}\right)(3b^2 + 4a\omega)}{4a^3\left(Q^2 - 3PR \pm Q\sqrt{Q^2 - 3PR}\right)}z + \omega t \tag{14}$$

Proof. Using the transformation (5), we reduce eq. (1) into an ODE of the form:

$$a^3cu^{(4)} + 6a^2cu'u'' + (4a\omega + 3b^2)u'' = 0 \tag{15}$$

Integrating eq. (15) once with respect to ξ and setting the integration constant as zero yields:

$$a^3cu''' + 3a^2cu'^2 + (4a\omega + 3b^2)u' = 0 \tag{16}$$

further letting $r = 1$ and $u' = v$, we have:

$$a^3cv'' + 3a^2cv^2 + (4a\omega + 3b^2)v = 0 \tag{17}$$

According to *Step 1*, we get $m + 2 = 2m$, hence $m = 2$. We then suppose that eq. (17) has a formal solution:

$$v(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi) \tag{18}$$

and substitute it into eq. (17), the following equation is obtained:

$$\begin{aligned}&3a_0b^2 + 3a^2a_0^2c + 4aa_0\omega + 3a_1b^2F(\xi) + 6a^2a_0a_1cF(\xi) + 4aa_1\omega F(\xi) + 3a_2b^2F^2(\xi) + \\ &+ 3a^2a_1^2cF^2(\xi) + 6a^2a_0a_2cF^2(\xi) + 4aa_2\omega F^2(\xi) + 6a^2a_1a_2cF^3(\xi) + \\ &+ 3a^3a_2^2cF^4(\xi) + 2a^3a_2cF'^2(\xi) + a^3a_1cF''(\xi) + 2a^3a_2cF(\xi)F''(\xi) = 0\end{aligned} \tag{19}$$

which gives the following equation by the use of eq. (9):

$$\begin{aligned}&3a_0b^2 + 3a^2a_0^2c + 2a^3a_2cR + 4aa_0\omega + 3a_1b^2F(\xi) + 6a^2a_0a_1cF(\xi) + \\ &+ a^3a_1cQF(\xi) + 4aa_1\omega F(\xi) + 3a_2b^2F^2(\xi) + 3a^2a_1^2cF^2(\xi) + \\ &+ 6a^2a_0a_2cF^2(\xi) + 4a^3a_2cQF^2(\xi) + 4aa_2\omega F^2(\xi) + 6a^2a_1a_2cF^3(\xi) + \\ &+ 2a^3a_1cPF^3(\xi) + 3a^2a_2^2cF^4(\xi) + 6a^3a_2cPF^4(\xi) = 0\end{aligned} \tag{20}$$

from which we have:

$$F^0(\xi): 3a_0b^2 + 3a^2a_0^2c + 2a^3a_2cR + 4aa_0\omega = 0 \tag{21}$$

$$F^1(\xi): 3a_1b^2 + 6a^2a_0a_1c + a^3a_1cQ + 4aa_1\omega = 0 \tag{22}$$

$$F^2(\xi): 3a_2b^2 + 3a^2a_1^2c + 6a^2a_0a_2c + 4a^3a_2cQ + 4aa_2\omega = 0 \tag{23}$$

$$F^3(\xi): 6a^2a_1a_2c + 2a^3a_1cP = 0 \tag{24}$$

$$F^4(\xi): 3a^2a_2^2c + 6a^3a_2cP = 0 \tag{25}$$

Solving above set of algebraic eqs. (21)-(25) by use of MATHEMATICA yields:

$$a_2 = -2aP, \quad a_1 = 0, \quad c = \frac{\left(Q \pm \sqrt{Q^2 - 3PR}\right)(3b^2 + 4a\omega)}{4a^3\left(Q^2 - 3PR \pm Q\sqrt{Q^2 - 3PR}\right)}, \quad a_0 = -\frac{2}{3}a\left(Q \pm \sqrt{Q^2 - 3PR}\right) \tag{26}$$

we, therefore, have eqs. (13) and (14). The proof is end.

When we employ eqs. (13) and (14), many exact solutions of eq. (1) can be obtained. For example, selecting $P = -m^2$, $Q = 2m^2 - 1$, $R = 1 - m^2$, $F(\xi) = \text{cn}\xi$, we obtain Jacobi elliptic function solution of eq. (1):

$$u(\xi) = 2am^2\int cn^2\xi \text{d}\xi - \frac{2}{3}a\left(2m^2 - 1 \pm \sqrt{m^4 - m^2 + 1}\right)\xi + c_0 \tag{27}$$

with

$$\xi = ax + by + \frac{\left(2m^2 - 1 \pm \sqrt{m^4 - m^2 + 1}\right)(3b^2 + 4a\omega)}{4a^3\left(m^4 - m^2 + 1 \pm (2m^2 - 1)\sqrt{m^4 - m^2 + 1}\right)}z + \omega t \tag{28}$$

Results and discussion

In order to further observe the obtained previous solutions, we show the dynamical evolutions of solution (27) with (+) branch in fig. 1, where the parameters are selected as $a = 1$, $b = 1$, $\omega = 1$, and $c_0 = 1$. It is easy to see from fig. 1 that such a solution (27) with external linear function ξ possess a remarkable dynamical property: the solitary wave does not propagate in the horizontal direction as the traditional waves do.

Using the properties of Jacobi elliptic functions in the limits at $m \to 1$ and $m \to 0$ [21], we can also obtain some hyperbolic function solutions and trigonometric function solutions from previously obtained Jacobi elliptic function solutions. For example, when $m \to 1$, solution (27) becomes:

$$u(\xi) = 2a\int \text{sech}^2\xi \text{d}\xi - \frac{2}{3}a(1 \pm 1)\xi + c_0 = 2a\tanh\xi - \frac{2}{3}a(1 \pm 1)\xi + c_0 \tag{29}$$

with

$$\xi = ax + by + \frac{(1 \pm 1)(3b^2 + 4a\omega)}{4a^3(1 \pm 1)}z + \omega t \tag{30}$$

and when $m \to 0$, the Jacobi elliptic function solution determined by $P = 1/4$, $Q = (1 - 2m^2)/2$, $R = 1/4$, and $F(\xi) = ns\xi \pm cs\xi$ becomes:

$$u(\xi) = -\frac{1}{2}a\int(\csc\xi \pm \cot\xi)^2 \mathrm{d}\xi - \frac{1}{6}a(2\pm 1)\xi + c_0 \tag{31}$$

where

$$\xi = ax + by + \frac{(2\pm 1)(3b^2 + 4a\omega)}{a^3(1\pm 2)}z + \omega t \tag{32}$$

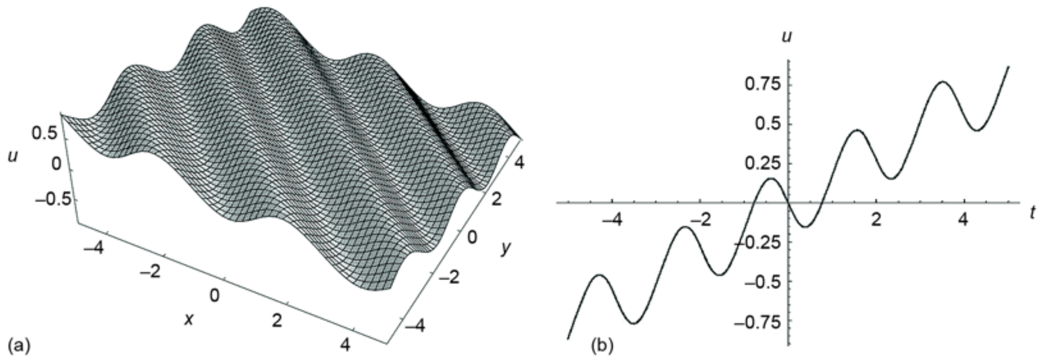

Figure 1. Dynamical evolutions of Jacobi elliptic function solution determined by eq. (27) with m = 0.6; (a) z = 0, t = 0, (b) x = 0, y = 0, z = 0

If we use Liu *et al.*'s method [11] to solve the (3+1)-D pYTSF eq. (1), usually we will balance u''' and u'^2 in eq. (16) and hence obtain $m + 3 = 2(m + 1)$, namely $m = 1$. We then suppose that eq. (16) has solution in the following form:

$$u(\xi) = a_0 + a_1 F(\xi) \tag{33}$$

and substitute it into eq. (16), the following equation can be obtained:

$$3a^2a_1^2cR + 3a^2a_1^2cQF^2(\xi) + 3a^2a_1^2cPF^4(\xi) + 3a_1b^2F'(\xi) + a^3a_1cQF'(\xi) +$$
$$+ 4aa_1\omega F'(\xi) + 6a^3a_1cPF^2(\xi)F'(\xi) = 0 \tag{34}$$

from which we have

$$3a^2a_1^2cR = 3a^2a_1^2cQ = 3a^2a_1^2cP = 3a_1b^2 + a^3a_1cQ + 4aa_1\omega = 6a^3a_1cP = 0 \tag{35}$$

and further obtain $a_1 = 0$ if considering the general case when $acPQR \neq 0$. In this case, we can only obtain trivial solution of eq. (1), this is not the one we expect.

It is obvious to see from previous comparison that all the Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions obtained in this paper can not be obtained by eq. (33) usually used in Liu *et al.*'s method [11] and its improvements [17-21] if we do not transform eq. (16) into eq. (17) but directly solve eq. (16). In this sense, we may conclude that the modified F-expansion method proposed in this paper is different from and superior to Liu *et al.*'s [11] method if and only if the reduced ODE (6) possesses property $r \geq 1$.

Remark 2. All the solutions previously obtained have been checked with MATHEMATICA by putting them back into the original eq. (1).

Conclusion

In summary, more general travelling wave solutions of the (3+1)-D pYTSF eq. (1) have been obtained owing to the modified F-expansion method proposed in this paper. The obtained Jacobi elliptic function solutions, hyperbolic function solutions, and trigonometric function solutions contain an explicit linear function of the variables in the pYTSF equation. It may be important to explain some physical phenomena. The paper shows the effectiveness and advantages of the proposed method beyond the original F-expansion method [11] and its existing improvements [17-21] in handing of the solution process of non-linear PDE. Employing the modified F-expansion method to solve some other non-linear PDE is our task in the future.

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