APPROXIMATE SOLUTION FOR FRACTIONAL BURGERS EQUATION WITH VARIABLE COEFFICIENT USING DAFTARDAR-GEJJI-JAFARIS METHOD

by

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A fractional Burgers equation with variable coefficients is studied, which can describe heat conduction in nanomaterials with intermittent property. The equation is solved analytically by Daftardar-Gejji-Jafaris method.

Key words: *fractional Burgers equations, variable coefficients, Caputo fractional derivative, Daftardar-Gejji-Jafaris method*

Introduction

In this paper, we study the approximate solution of a fractional Burgers' equation with variable coefficients of the type:

$$\mathrm{D}_t^\alpha u(x,t) + \lambda(x,t)u(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \qquad 0 < \alpha \le 1 \tag{1}$$

with the initial condition:

$$u(x,t) = f(x) \tag{2}$$

Here, $\lambda(x, t)$ is the given function. Equation (1) arises in the mathematical modeling of various physical phenomena, such as heat exchange in nanomaterials [1-12]. Moreover, the Burgers' equation with variable coefficient can be used to describe the cylinder and spherical wave in overfall, and traffic flow, see for example [8, 13, 14]. The time-fractional term in eq. (1) implies that at the measured time period, time is discontinuous and it has intermittent property, for example, a traffic flows at the daytime and at the night-time have obvious difference. When $\alpha = 1$ the traffic flow has the same properties throughout the whole day, when $\alpha = 0$ the traffic flow does not change with time. Such intermittent motion can be best described by the time-fractional model.

In eq. (1):

$$\mathrm{D}_t^\alpha u(x,t) = \mathrm{J}_t^{1-\alpha}\left[\frac{\partial u(x,t)}{\partial t}\right] \tag{3}$$

is the Caputo fractional derivative of order α, J_t^μ denotes the Riemann-Liouville fractional integral operator of order $\mu \ge 0$ and is given by:

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$$\mathrm{J}_t^{\mu} u(x,t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} u(x,s)\,\mathrm{d}s \qquad \mu > 0 \tag{4}$$

$$\mathrm{J}_t^{0} u(x,t) = u(x,t)$$

The following properties can be found in [26, 27]:

$$\mathrm{J}_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)} t^{\gamma+\alpha}, \qquad \alpha \geq 0, \quad \gamma > -1 \tag{5}$$

$$\mathrm{J}_t^{\alpha} \mathrm{D}_t^{\alpha} u(x,t) = u(x,t) - u(x,0), \qquad 0 < \alpha < 1 \tag{6}$$

In the last two decades, some numerical and analytical methods for solving fractional differential equations have been extensively studied by many authors [18-25]. The Daftardar-Gejji-Jafaris (DGJ) method was proposed by Daftardar-Gejji, Varsha, and Hossein Jafari in [15, 16]. It is a powerful tool to searching for approximate solutions of non-linear problems. Recently, Daftardar-Gejji, Varsha and Sachin Bhalekar [17] found the exact solution and approximate solution of fractional differential equations using DGJ method.

The DGJ method

To illustrate the DGJ method (DGJM) [15, 16], we consider the following general function equation:

$$u = L(u) + N(u) + f \tag{7}$$

where L is a linear operator, N – a non-linear operator from a Banach space $B \rightarrow B$, and f – a known function. We are looking for a solution u of eq. (7) having the series form:

$$u = \sum_{i=0}^{\infty} u_i \tag{8}$$

The non-linear operator N can be decomposed:

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{\infty} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{9}$$

From eqs. (8) and (9), eq. (7) is equivalent to:

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + N(u_0) + \sum_{i=1}^{\infty}\left[N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right] \tag{10}$$

We define the recurrence relation:

$$u_0 = f(x)$$

$$u_1 = L(u_0) + G_0$$

$$u_m = L(u_m) + G_m, \quad m = 1, 2... \tag{11}$$

where

$$G_0 = N(u_0) \tag{12}$$

$$G_m = N\left(\sum_{i=0}^{m} u_i\right) - N\left(\sum_{i=0}^{m-1} u_i\right), \quad m = 1, 2... \tag{13}$$

Then k-term approximate solution of eq. (7) is given by:

$$u = u_0 + u_1 + ... + u_{k-1} \tag{14}$$

Fractional Burgers equation

In this section we derive the main algorithms of the DGJM for solving fractional Burgers equations with variable coefficients.

To apply DGJM, by eq. (6), we can rewrite the eq. (1):

$$u(x,t) = u(x,0) + L(u) - N(u) \tag{15}$$

where

$$L(u) = \mathrm{J}_t^{\alpha}\left(\frac{\partial^2 u}{\partial x^2}\right)$$

$$N(u) = \mathrm{J}_t^{\alpha}\left[\lambda(x,t) u \frac{\partial u}{\partial x}\right] \tag{16}$$

Suppose that the solution of eq. (15) takes the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \tag{17}$$

then

$$L(u) = \sum_{k=0}^{\infty} L(u_k) = \sum_{k=0}^{\infty} \mathrm{J}_t^{\alpha}\left(\frac{\partial^2 u_k}{\partial x^2}\right)$$

and the non-linear term in eq. (15) is decomposed:

$$N(u) = \mathrm{J}_t^{\partial}\left[\lambda(x,t) G_0(u_0) + G_1(u_0,u_1) + G_2(u_0,u_1,u_2) + ...\right] \tag{18}$$

Where

$$G_0(u_0) = u_0 \frac{\partial u_0}{\partial x}$$

$$G_1 = (u_0, u_1) = (u_0 + u_1)\frac{\partial (u_0 + u_1)}{\partial x} - u_0 \frac{\partial u_0}{\partial x}$$

$$G_2(u_0,u_1,u_2) = (u_0,u_1,u_2)\frac{\partial (u_0,u_1,u_2)}{\partial x} - (u_0 + u_1)\frac{\partial (u_0 + u_1)}{\partial x}$$

and so on.

Thus according to eq. (11), approximate solution can be obtained:

$$u_0 = (x,t) = u(x,0)$$

$$u_1(x,t) = \mathrm{J}_t^{\alpha}\left(\frac{\partial^2 u_0}{\partial x^2}\right) + \mathrm{J}_t^{\alpha}\left[\lambda(x,t) G_0\right] \tag{19}$$

$$u_{m+1}(x,t) = \mathrm{J}_t^{\alpha}\left(\frac{\partial^2 u_m}{\partial x^2}\right) + \mathrm{J}_t^{\alpha}\left[\lambda(x,t)G_m\right], \quad m = 1,2,...$$

For example, we consider eq. (1) in the form:

$$\mathrm{D}_t^{\alpha} u(x,t) + tu(x,t)\frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = 0 \tag{20}$$

with the initial condition $u(x, 0) = x$.

By the previous algorithms, we obtain:

$$u_0(x,t) = x^2$$

$$u_1(x,t) = \frac{2t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2x^3 t^{1+\alpha}}{\Gamma(2+\alpha)}$$

$$u_2(x,t) = \frac{10x^4(2+\alpha)t^{2+2\alpha}}{\Gamma(3+2\alpha)} + \frac{12x^2\Gamma(3+2\alpha)t^{2+3\alpha}}{\Gamma(1+\alpha)\Gamma(2+\alpha)\Gamma(3+3\alpha)} - \\ - \frac{4x(1+\alpha)t^{1+2\alpha}}{\Gamma(2+2\alpha)} - \frac{12x^5\Gamma(4+2\alpha)t^{3+3\alpha}}{\Gamma^2(2+\alpha)\Gamma(4+3\alpha)}$$

Thus, the 3-*term* approximate solution of eq. (20) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t)$$

The accuracy of the DGJM based on absolute error (AE) are shown in tabs. 1 and 2.

Table 1. u **= 6 – term approximate solution of eq. (20).** $AE = |\mathrm{D}_t^{\alpha} u + tuu_x - u_{xx}|$

t	x	α	AE
0.5	0.5	0.5	$0.054286578 \cdot 10^{-4}$
	0.7		$0.150004787 \cdot 10^{-4}$
	0.9		$0.596447498 \cdot 10^{-4}$

Table 2. u **= 6 – term approximate solution of eq. (20).** $AE = |\mathrm{D}_t^{\alpha} u + tuu_x - \mu_{xx}|$

x	t	α	AE
0.2	0.2	0.5	$0.051655319 \cdot 10^{-4}$
	0.4		$0.044552560 \cdot 10^{-4}$
	0.6		$0.645009601 \cdot 10^{-4}$

Conclusion

We presented the application of DGJM to a fractional Burgers' equations with variable coefficients. The DGJM gives series solutions of the equation. Compared to the other methods, the DGJM is direct and effective. Furthermore the solution reveals that the intermittent property depends upon the value of the fractional order.

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