EFFETS OF CNT$_S$ ON MAGNETOHYDRODYNAMIC FLOW OF METHANOL BASED NANOFLUIDS VIA ATANGANA-BALEANU AND CAPUTO-FABRIZIO FRACTIONAL DERIVATIVES

by

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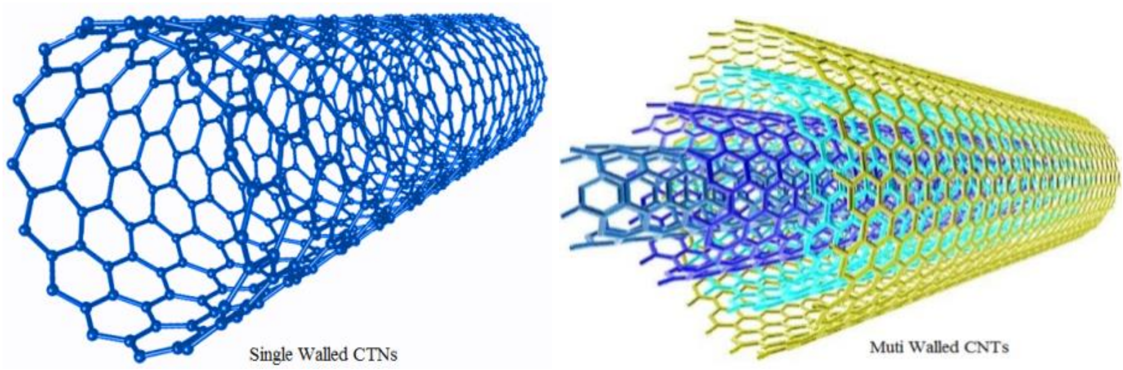
*This paper dedicatedly reports the heat transfer analysis of single and multi-walls carbon nanotubes (SWCNTs and MWCNTs) for electrically conducting flow of Casson fluid. Both types of carbon nanotubes are suspended in methanol that is considered as a conventional base fluid. The governing partial differential equations of nanofluids have been modeled by employing newly defined fractional approaches (derivatives) namely Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established via both approaches i-e (AB) and (CF) fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}(T)$ and generalized **M**-function $\mathbf{M}_{q}^{p}(F)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases (AB and CF fractional derivatives) are depicted for graphical illustrations. The comparison of three types of fluids (i) pure methanol (ii) methanol with single walls carbon nanotubes and (iii) methanol with multi-walls carbon nanotubes is portrayed via Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. Finally, the results indicate that, pure methanol moves quicker in comparison with methanol-SWCNTs via Caputo-Fabrizio (CF)and methanol-MWCNTs, while for larger time, methanol-MWCNTs moves more rapidly in comparison with pure methanol and methanol-SWCNTs via Atangana-Baleanu (AB).*

Key words: *Casson fluid, Nanoparticles, MHD, fractional derivatives.*

Introduction

The analysis of non-Newtonian liquids has attained significant consideration due to their immersion in extensive engineering and industrial applications. Such applications immerse plastics manufacturing, petroleum production, bioengineering, drawing of stretching sheet through quiescent fluid, food processing, polymeric liquids, aerodynamic extrusion of plastic films, annealing and thinning of copper wires and several others. There is no denying fact that the Navier-Stokes expression does not characterize flows of non-Newtonian liquids. A single relation is not adequate to predict the characteristics and features of non-Newtonian liquids, even numerous non-Newtonian relations are available in vast literature [1-5]. Among the categories of non-Newtonian liquids, a Casson fluid is one of them. A Casson fluid has proved to be the best for the description of shear-thinning liquids having zero viscosity at an infinite rate of shear and infinite viscosity at zero shear rates [6-7]. In continuation, the development of industrial manufacturing processes totally depends upon greater heat transfer rates. The normal techniques for heat transfer are not adequate to supply reasonable heat transfer rates for industrial and manufacturing needs. In order to develop the technique for enhancing heat transfer, several scientists, mathematicians, engineers and numerical analysts are working in this field. Their main purpose is to enhance the performance of various liquids for instance, water, ethylene-glycol, oil and few others. The abrasion, clogging, additional pressure loss etc. are the aspects which are inadequate to enhance thermal conductivity. The idea of nanofluids was proposed by Choi, he verified through his experimental work that thermal performance of carrier-liquid can be enhanced via merging/submersion of (tiny) small size metallic or solid particles [8]. The mechanism of nanofluids via nanoparticles in base fluid was investigated by Buongiorno [9]. In this mechanism, nanoparticles include size, inertia, magnus effect, Brownian motion, particle agglomeration, volume fraction and thermophoresis. On the other hand, nanofluids are utilized to enhance heat transfer rate and thermal conductivity of base fluids [10-18]. For this purpose, different researchers utilize various types of nanoparticles with distinct sizes and shapes. Most the nanoparticles are made up of oxides, metals, carbon nanotubes (CNTs), carbides etc. while engine oil, kerosene, ethylene glycol and water are considered as the base fluids. Carbon nanotubes are mainly divided into two categories: (i) SWCNTs (single walls carbon nanotubes) and (ii) MWCNTs (multiple walls carbon nanotubes) as in below [67] Fig. 1



Single and Multi Walled CNTs.

In brevity, carbon nanotubes have gotten significant attention due to highest thermal conductivity in comparison with other nanostructure materials. The thermal conductivity of SWCNTs and MWCNTs is substantially higher than the thermal conductivity of metal oxide nanoparticles or metal nanoparticles. The SWCNTs and MWCNTs have remarkable thermal, optical, electrical and mechanical properties due to cylindrical carbon molecules origin. Hence, they have been declared as best tools for special thermal properties having extra thermal conductivities, this is due to the facts enumerated as (i) The diameter of SWCNTs and MWCNTs which ranges between 1 to 100 nm while length in micrometer (ii) The SWCNTs and MWCNTs have two hundred times strength (iii) The SWCNTs and MWCNTs have fifteen times thermal conductivity (iv) The SWCNTs and MWCNTs have hundred times current capacity of copper (v) The SWCNTs and MWCNTs have five times elasticity of steel (vi) The SWCNTs and MWCNTs have high aspect ratio which assist to form a network of conductive tubes (vii) The mechanical and electronic properties of SWCNTs and MWCNTs can be implemented for instance, leading-edge electronic fabrication, field-emission displays, nanosensors, nanocomposite materials. Even this, Environmental Protection Agency (EPA) has declared the SWCNTs and MWCNTs as the non-hazardous particles for the environment. However, in comparison with various other nanoparticles, several researchers are engaged to work on the SWCNTs and MWCNTs for the enhancement of thermal conductivity. Liu *et al.* [19] analyzed the single and multi-walled carbon nanotubes for enhancing the thermal conductivity of engine oil and ethylene glycol. They inspected that ethylene glycol fluid without CTNs have lower thermal conductivities in comparison with ethylene glycol fluid with CTNs. Thermal conductivity of ethylene glycol fluid with CTNs at volume fraction of 0.01 was increased by 12.4%, in contrast to this, thermal conductivity of engine oil with CTNs at volume fraction of 0.02 was increased by 30%. Marquis *et al.* [20] examined the improvement in the thermal conductivity of SWCNTs and MWCNTs with nanolubricants and nanofluids, in which they considered three distinct types of nanolubricants and nanofluids. Xie *et al.* [21] inspected nanofluids consisting of MWCNTs for enhancing the thermal conductivities experimentally. They checked that increment in CNTs into fluid generates enhancement in thermal conductivity. Khan *et al.* [22] analyzed the fluid problem with Navier slip boundary condition on heat transfer of CNTs over flat plate. They investigated that engine oil and kerosene based CNTs have lower thermal conductivities and densities in comparison with water based CNTs. Haq *et al.* [23] examined SWCNTs and MWCNTs for water based flow; they perceived lower Nusselt number and skin friction for MWCNTs in comparison with SWCNTs. In the similar study, Haq *et al.* [24] investigated kerosene and water based CNTs fluid has lower heat transfer and skin friction in comparison with engine oil based CNTs fluid. Camilli *et al.* [25] presented experimental study of viscosity of CNTs for water based nanofluids, in which they considered the impacts of volume fraction and temperature. Kamali *et al.* [26] explored numerical study for MWCNTs with nanofluids using fixed wall heat flux condition in a straight tube. They observed that due to non-Newtonian behavior of CNTs nanofluids, heat transfer coefficient is dominated by wall region. Of course, the list of studies on SWCNTs and MWCNTs, magnetohydrodynamics, fractional derivatives, porous medium and nanoparticles with nanofluids [27-34] can be carried on, but we close it with some of the most interesting references published recently [35-46]. In the area of fluid mechanics and computational fluid dynamics, the fractional derivatives of non-integer orders are well-known. There are several types of fractional derivatives. These fractional derivatives have been recommended by prominent mathematicians as Riemann, Liouville, Sonin, Weyl, Letnikov, Erdelyi, Riesz, Kober and many others. The fractional derivatives of non-integer orders are enthusiastically used in the fluid mechanics, computational fluid dynamics and natural sciences to evaluating the systems and processes with spatial and temporal nonlocality (the nonlocality in time is usually called memory). In this continuity, Saqib *et al.* [47] investigated free convection flow of Jeffrey fluid by employing Caputo-Fabrizio time-fractional derivatives and recovered the existing solutions in the open literature as well. Nadeem *et al.* [48] presented an interesting comparative analysis via Caputo and Fabrizio and Atangana and Baleanu fractional derivatives based on exponential and generalized Mittag-Leffler function as a kernel. They implemented the non-singular and non-local kernel on free convection flow of a generalized Casson fluid. Nadeem *et al.* [49] implemented modern fractional derivatives with the non-singular and non-local kernel to enhance the heat transfer rate of solar energy devices via nanoparticles. Aftab *et al.* [50] analyzed Molybdenum Disulphide (MoS_2) nanoparticles of spherical shape and suspended in engine oil based generalized Brinkman-type nanofluid via newly introduced fractional derivatives known as Atangana-Baleanu Derivative. Motivating by above research study, our aim is to report the heat transfer analysis of single and multi-walls carbon nanotubes (SWCNTs and MWCNTs) for electrically conducting flow of Casson fluid. Both types of carbon nanotubes are suspended in methanol which is considered as a conventional base fluid. The governing partial differential equations of nanofluids have been modeled by employing newly defined fractional derivatives namely Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established via both approaches i-e (AB) and (CF) fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}^{y}_{\varepsilon,\delta}(T)$ and generalized **M**-function $\mathbf{M}^{p}_{q}(F)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases (AB and CF fractional derivatives) are depicted for graphical illustrations. Finally, the comparison of three types of fluids (i) pure methanol (ii) methanol with single walls carbon nanotubes and (iii) methanol with multi-walls carbon nanotubes is portrayed via Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives.

Governing Equations

Let us consider an incompressible unsteady flow of Casson-nanofluid containing CNTs occupying a semi-finite space $y > 0$. The plate is assumed to be electrically conducting with a uniform magnetic field B of strength B_0, applied in a direction perpendicular to the plate. The magnetic Reynolds number is assumed to be small enough to neglect the effect of applied magnetic field. CNTs are suspended in methanol taken as base fluid. Initially, at time $t = 0,$ both the fluid and the plate are at rest with constant temperature T_∞. At time $t = 0^+$, the plate is subjected to accelerated motion. At the same time, the plate temperature is raised to T_w which is thereafter maintained constant. Following [48-50], the equations governing the flow and heat transfer are given by

$$\rho_{nf}\frac{\partial F(y,t)}{\partial t} = \mu_{nf}\left(\frac{1}{\beta}+1\right)\frac{\partial^2 F(y,t)}{\partial y^2} - \sigma_{nf}B_0^2 F(y,t) + g\left(T-T_\infty\right)\left(\rho\beta_T\right)_{nf}, \tag{1}$$

$$\left(\rho c_p\right)_{nf}\frac{\partial T(y,t)}{\partial t} = k_{nf}\frac{\partial^2 T(y,t)}{\partial y^2}, \tag{2}$$

where, μ_{nf} is the dynamic viscosity, β is Casson fluid parameter, ρ_{nf} is density, σ_{nf} is electrical conductivity, g gravitational acceleration, $(\rho\beta)_{nf}$ volumetric thermal expansion coefficient, $(\rho c_p)_{nf}$ heat capacitance and k_{nf} thermal conductivity. The subscript (nf) is used for nanofluid whereas (f) will be used for base fluid.

Initial and boundary conditions are:

$$\begin{aligned} &F(y,0) = 0, \quad T(y,0) = T_\infty, \qquad y \ge 0,\\ &F(0,t) = At^p, \quad T(0,t) = T_w, \qquad t > 0,\\ &F(\infty,t) = 0, \quad T(\infty,t) = T_\infty, \qquad t > 0. \end{aligned} \tag{3}$$

Here, A is arbitrary constant and its dimension depends on the value of (p) upon t. For constantly accelerated motion dimension of A will be $[L/T^2]$ and for variably accelerated motion, dimension of A will be $[L/T^3]$.

Using Xue [57] model for dynamic viscosity and thermal conductivity as:

$$\mu_{nf}/\mu_f = 1/(1-\phi)^{2.5},$$

$$k_{nf}/k_f = 2\phi\left(k_{CNTs}/k_{CNTs}-k_f\right)\ln\left(\frac{k_f+k_{CNTs}}{2k_f}\right)-\phi+1\Big/2\phi\left(k_f/k_{CNTs}-k_f\right)\ln\left(\frac{k_f+k_{CNTs}}{2k_f}\right)-\phi+1, \quad (4)$$

The ρ_{nf}, σ_{nf}, $(\rho\beta)_{nf}$ and $(\rho c_p)_{nf}$ are given by:

$$\begin{aligned}
\rho_{nf} &= \phi\rho_{CNTs}+\rho_f(1-\phi),\\
\sigma_{nf} &= \left[1+\frac{\phi(3\sigma_{CNTs}-3)}{(2+\sigma_{CNTs})-\phi(\sigma_{CNTs}-1)}\right]\sigma_f,\\
(\rho\beta)_{nf} &= \phi(\rho\beta)_{CNTs}+(\rho\beta)_f(1-\phi),\\
(\rho c_p)_{nf} &= \phi(\rho c_p)_{CNTs}+(\rho c_p)_f(1-\phi),
\end{aligned} \quad (5)$$

Introducing the following dimensionless variables:

$$t^* = \frac{tU_0^2}{\nu}, \quad F^* = \frac{F}{U_0}, \quad y^* = \frac{yU_0}{\nu}, \quad T^* = \frac{T-T_\infty}{\Delta T}; \ \Delta T = T_w - T_\infty, \quad (6)$$

into equations (1-4), one gets:

$$a_0\frac{\partial F(y,t)}{\partial t} = a_1\frac{\partial^2 F(y,t)}{\partial y^2} - a_2F(y,t) + a_3T(y,t), \quad (7)$$

$$\frac{\partial T(y,t)}{\partial t} = a_4\frac{\partial^2 T(y,t)}{\partial y^2}. \quad (8)$$

Here

$$\begin{aligned}
&a_0 = \phi_1 = 1-\phi+\phi\frac{\rho_{CTNs}}{\rho_f},\ a_1 = \phi_2\left(1+\frac{1}{\beta}\right),\ a_2 = M\phi_3,\ a_3 = Gr\phi_3,\ a_4 = \frac{\lambda_{nf}}{\Pr\phi_5}\\
&M = \frac{\sigma_f B^2{}_0\nu_f}{U^2{}_0\rho_f},\ G_r = \frac{\nu_f g\Delta T\beta_f}{U_0^3},\ \lambda_{nf} = \frac{k_{nf}}{k_f},\ \Pr = \frac{(\mu c_p)_f}{k_f}\\
&\phi_2 = \frac{1}{(1-\phi)^{2.5}},\ \phi_3 = 1+\frac{3\phi(\sigma_{CNTs}-1)}{(2+\sigma_{CNTs})-\phi(\sigma_{CNTs}-1)}\\
&\phi_4 = 1-\phi+\frac{\phi(\rho\beta_T)_{CTN}}{(\rho\beta_T)_f},\ \phi_5 = 1-\phi+\frac{\phi(\rho c_p)_{CNTs}}{(\rho c_p)_f}.
\end{aligned} , \quad (9)$$

The initial and boundary conditions become:

$$\begin{aligned}
&F(y,0)=0, \quad T(y,0)=0, \quad y\ge 0, \quad F(0,t)=At^p, \quad T(0,t)=1, \quad t>0,\\
&F(\infty,t)=0, \quad T(\infty,t)=0, \quad , \quad t>0.
\end{aligned} \quad (10)$$

Calculation of Problem

Calculation of Temperature and Velocity Fields via Atangana-Baleanu Fractional Derivatives

For developing Atangana-Baleanu fractional model, the governing equations (7-8) have been replaced by time variable of order α equations (7-8) are formulated in terms of Atangana-Baleanu fractional operator as:

$$^{AB}\left(\frac{\partial^\alpha F(y,t)}{\partial t^\alpha}\right) - \frac{a_1}{a_0}\frac{\partial^2 F(y,t)}{\partial y^2} + \frac{a_2}{a_0}F(y,t) + \frac{a_3}{a_0}T(y,t) = 0, \quad (11)$$

$$ {}^{AB}\left(\frac{\partial^{\alpha}T(y,t)}{\partial t^{\alpha}}\right)-a_4\frac{\partial^2 T(y,t)}{\partial y^2}=0, \tag{12} $$

Where, ${}^{AB}\left(\frac{\partial^{\alpha}F(y,t)}{\partial t^{\alpha}}\right)$ is Atangana-Baleanu fractional differential operator of order α defined as [60-62]

$$ {}^{AB}\left(\frac{\partial^{\alpha}F(y,t)}{\partial t^{\alpha}}\right)=\frac{M(\alpha)}{1-\alpha}\int_0^t F'(y,t)\mathbf{E}_{\alpha}\left(\frac{-\alpha(z-t)^{\alpha}}{1-\alpha}\right)dt, \tag{13} $$

Here $M(\alpha)$ is normalization function such that $M(0)=M(1)=1$. Employing Laplace transform on equations (11-12) and imposing assumed conditions ($10_{1,2,3}$), taking $\chi=\frac{1}{1-\alpha}$, we obtain

$$ \left(\frac{\partial^2}{\partial y^2}-\frac{a_0\,\chi\, s^{\alpha}}{a_1\,(s^{\alpha}+\alpha\chi)}-\frac{a_2}{a_1}\right)\bar{F}(y,s)=\frac{a_3}{a_1}\bar{T}(y,s), \tag{14} $$

$$ \left(\frac{\partial^2}{\partial y^2}-\frac{s^{\alpha}\chi}{a_4\left(s^{\alpha}+\alpha\chi\right)}\right)\bar{T}(y,s)=0, \tag{15} $$

Here, $\bar{F}(y,s)$ and $\bar{T}(y,s)$ are the Laplace transform of $F(y,t)$ and $T(y,t)$, computing equations (14-15), we arrive

$$ \bar{F}(y,s)=\frac{Ap!}{s^{p+1}}Exp\left(-y\sqrt{\frac{a_0\,\chi\, s^{\alpha}}{a_1\,(s^{\alpha}+s_2)}-\frac{a_2}{a_1}}\right)+\frac{a_3\,(s^{\alpha}+s_2)}{s(s_3 s^{\alpha}-s_4)}Exp\left(-y\sqrt{\frac{s_1 s^{\alpha}}{(s^{\alpha}+s_2)}}\right), \tag{16} $$

$$ \bar{T}(y,s)=\frac{1}{s}Exp\left(-y\sqrt{\frac{s_1\, s^{\alpha}}{(s^{\alpha}+s_2)}}\right), \tag{17} $$

Where, $s_1=\chi/a_4$, $s_2=\alpha\chi$, $s_3=a_0a_4\chi-a_2a_4+a_1\chi$, $s_4=a_2a_4\alpha\chi$.
In order to obtain velocity and temperature profiles, we write equations (16-17) into series form, we get equivalent expressions as

$$ \begin{aligned} \bar{F}(y,s)=&\frac{Ap!}{s^{p+1}}+\frac{Ap!}{s^{p+1}}\sum_{p_1=1}^{\infty}\frac{1}{p_1\,!}\left(-y\sqrt{\frac{s_5}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2\,!}\left(-\frac{s_6}{s_5}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{1}{p_3\,!}\left(-s_2\right)^{p_3}\sum_{p_4=0}^{\infty}\left(-\frac{s_3}{s_4}\right)^{p_4}\\ &\times\frac{\Gamma\left(1+\frac{p_1}{2}\right)\Gamma\left(p_3+\frac{p_1}{2}\right)}{\Gamma\left(\frac{p_1}{2}-p_2+1\right)\Gamma\left(\frac{p_1}{2}\right)s^{\alpha p_2+\alpha p_3-\alpha p_4}}-\frac{a_3}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1\,!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{S_4}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_2}\right)^{p_3}\Gamma\left(p_3+\frac{p_1}{2}\right)}{p_3\,!\Gamma\left(\frac{p_1}{2}\right)s^{\alpha p_3-\alpha p_2-\alpha+1}}\\ &-\frac{a_3s_2}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1\,!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{s_4}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_2}\right)^{p_3}\Gamma\left(p_3+\frac{p_1}{2}\right)}{p_3\,!\Gamma\left(\frac{p_1}{2}\right)s^{\alpha p_3-\alpha p_2-\alpha+1}}, \end{aligned} \tag{18} $$

$$ \bar{T}(y,s)=\frac{1}{s}+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1\,!}\sum_{p_2=0}^{\infty}\frac{\left(-s_2\right)^{p_2}\Gamma\left(p_2+\frac{p_1}{2}\right)}{p_2\,!\Gamma\left(\frac{p_1}{2}\right)s^{\alpha p_2+1}}, \tag{19} $$

Where, $s_5=a_0\chi-a_2$, $s_6=-a_2s_2$. Inverting equations (18-19) via Laplace transform and expressing the final solutions in terms of newly defined **M**-function $\mathbf{M}_q^p(F)$ and Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}(T)$ respectively as

$$F(y,t)=At^{p}+Ap!\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_5}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_6}{s_5}\right)^{p_2}\sum_{p_4=0}^{\infty}\left(-\frac{s_3}{s_4}\right)^{p_4}$$

$$\times\mathbf{M}_3^2\left[\left(-s_2\right)\left|\begin{matrix}\left(\frac{p_1}{2}+1,0\right),\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\left(\frac{p_1}{2}-p_2+1,0\right),\left(\alpha p_2-\alpha p_4-1,\alpha\right)\end{matrix}\right.\right]$$

$$-\frac{a_3}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{S_4}\right)^{p_2}\mathbf{M}_2^1\left[\left(-s_2\right)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\left(-\alpha p_2-\alpha,\alpha\right)\end{matrix}\right.\right]$$

$$-\frac{a_3 s_2}{s_4}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_3}{S_4}\right)^{p_2}\mathbf{M}_2^1\left[\left(-s_2\right)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\left(-\alpha p_2-\alpha,\alpha\right)\end{matrix}\right.\right]. \tag{20}$$

$$T(y,t)=1+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_1}\right)^{p_1}}{p_1!}\,\mathbf{M}_{p_2,1}^{\frac{p_1}{2}}\left(-s_2 t\right). \tag{21}$$

Where, the newly defined Generalized **M**-function $\mathbf{M}_q^p(F)$ [63] and Mittage Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}(T)$ are described respectively

$$t^{\varepsilon_q-1}\sum_{\delta}^{\infty}\frac{(F)^{\delta}\prod_{h=1}^{f}\Gamma(a_h+A_h\delta)}{\delta!\prod_{h=1}^{g}\Gamma(b_h+B_h\delta)}=\mathbf{M}_q^p\left[F\left|\begin{matrix}(a_1,A_1),(a_2,A_2),\ldots,(a_f,A_f)\\(b_1,B_1),(b_2,B_2),\ldots,(b_g,B_g)\end{matrix}\right.\right]. \tag{22}$$

$$t^{\delta-1}\sum_{\zeta}^{\infty}\frac{(T)^{\zeta}\Gamma(y+\zeta)}{\zeta!\Gamma(y)\Gamma(\varepsilon\zeta+\delta)}=t^{\delta-1}\mathbf{E}_{\varepsilon,\delta}^{y}(T)=\mathbf{M}_{\varepsilon,\delta}^{y}(T),\qquad \mathrm{Re}(\varepsilon)>0,\quad \mathrm{Re}(\delta)>0, \tag{23}$$

Calculation of Temperature and Velocity Fields via Caputo-Fabrizio Fractional Derivatives

For developing Caputo-Fabrizio fractional model, the governing equations (7-8) have been replaced by time variable of order β equations (7-8) are formulated in terms of Caputo-Fabrizio fractional operator as:

$$^{CF}\left(\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right)-\frac{a_1}{a_0}\frac{\partial^2F(y,t)}{\partial y^2}+\frac{a_2}{a_0}F(y,t)+\frac{a_3}{a_0}T(y,t)=0, \tag{24}$$

$$^{CF}\left(\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right)-a_4\frac{\partial^2T(y,t)}{\partial y^2}=0, \tag{25}$$

Where, $^{CF}\left(\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right)$ is the Caputo-Fabrizio fractional differential operator of order β defined as [64-66]

$$^{CF}\left(\frac{\partial^{\beta}F(y,t)}{\partial t^{\beta}}\right)=\frac{M(\beta)}{1-\beta}\int_0^t F'(y,t)\,Exp\left(\frac{-(t-\delta)\beta}{(1-\beta)}\right)dt. \tag{26}$$

Here $M(\beta)$ is normalization function such that $M(0)=M(1)=1$. Employing Laplace transform on equations (24-25) and imposing assumed conditions ($10_{1,2,3}$), taking $\tau=\frac{1}{1-\beta}$, we obtain

$$\left(\frac{\partial^2}{\partial y^2}-\frac{a_0\tau s}{a_1(s+\beta\tau)}-\frac{a_2}{a_1}\right)\bar{F}(y,s)=\frac{a_3}{a_1}\bar{T}(y,s), \tag{27}$$

$$\left(\frac{\partial^2}{\partial y^2}-\frac{s\tau}{a_4\left(s+\tau\beta\right)}\right)\bar{T}(y,s)=0, \tag{28}$$

Here, $\bar{F}(y,s)$ and $\bar{T}(y,s)$ are the Laplace transform of $F(y,t)$ and $T(y,t)$, computing equations (27-28), we arrive

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}Exp\left(-y\sqrt{\frac{a_0\tau s}{a_1(s+s_8)}-\frac{a_2}{a_1}}\right)+\frac{a_3(s+s_8)}{s(s_9s-s_{10})}Exp\left(-y\sqrt{\frac{s_7s}{(s+s_8)}}\right), \tag{29}$$

$$\bar{T}(y,s)=\frac{1}{s}Exp\left(-y\sqrt{\frac{s_7s}{(s+s_8)}}\right), \tag{30}$$

Where, $s_7=\tau/a_4$, $s_8=\beta\tau$, $s_9=a_0a_4\tau-a_2a_4+a_1\tau$, $s_{10}=a_2a_4\beta\tau$. In order to obtain velocity and temperature profiles, we write equations (29-30) into series form, we get equivalent expressions as

$$\bar{F}(y,s)=\frac{Ap!}{s^{p+1}}+\frac{Ap!}{s^{p+1}}\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_{11}}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_{12}}{s_{11}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{1}{p_3!}\left(-s_8\right)^{p_3}\sum_{p_4=0}^{\infty}\left(-\frac{s_9}{s_{10}}\right)^{p_4}$$
$$\times\frac{\Gamma\left(1+\frac{p_1}{2}\right)\Gamma\left(p_3+\frac{p_1}{2}\right)}{\Gamma\left(\frac{p_1}{2}-p_2+1\right)\Gamma\left(\frac{p_1}{2}\right)s^{p_2+p_3-p_4}}-\frac{a_3}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{S_{10}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_8}\right)^{p_3}\Gamma\left(p_3+\frac{p_1}{2}\right)}{p_3!\Gamma\left(\frac{p_1}{2}\right)s^{p_3-p_2}}$$
$$-\frac{a_3s_8}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\sum_{p_3=0}^{\infty}\frac{\left(\sqrt{s_8}\right)^{p_3}\Gamma\left(p_3+\frac{p_1}{2}\right)}{p_3!\Gamma\left(\frac{p_1}{2}\right)s^{p_3-p_2}}, \tag{31}$$

$$\bar{T}(y,s)=\frac{1}{s}+\sum_{p_1=1}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\frac{\left(-s_8\right)^{p_2}\Gamma\left(p_2+\frac{p_1}{2}\right)}{p_2!\Gamma\left(\frac{p_1}{2}\right)s^{p_2+1}}, \tag{32}$$

Where, $s_{11}=a_0\tau-a_2$, $s_{12}=-a_2s_8$. Inverting equations (31-32) via Laplace transform and expressing the final solutions in terms of newly defined **M**-function $\mathbf{M}_q^p(F)$ and Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^y(T)$ respectively as

$$F(y,t)=At^p+Ap!\sum_{p_1=1}^{\infty}\frac{1}{p_1!}\left(-y\sqrt{\frac{s_{11}}{a_1}}\right)^{p_1}\sum_{p_2=0}^{\infty}\frac{1}{p_2!}\left(-\frac{s_{12}}{s_{11}}\right)^{p_2}\sum_{p_4=0}^{\infty}\left(-\frac{s_9}{s_{10}}\right)^{p_4}$$
$$\times\mathbf{M}_3^2\left[\left(-s_8\right)\left|\begin{matrix}\left(\frac{p_1}{2}+1,0\right),\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\left(\frac{p_1}{2}-p_2+1,0\right),\left(p_2-p_4-1,1\right)\end{matrix}\right.\right]$$
$$-\frac{a_3}{s_{10}}\sum_{p_1=0}^{\infty}\frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1!}\sum_{p_2=0}^{\infty}\left(\frac{s_9}{s_{10}}\right)^{p_2}\mathbf{M}_2^1\left[\left(-s_8\right)\left|\begin{matrix}\left(\frac{p_1}{2},1\right)\\ \left(\frac{p_1}{2},0\right),\left(-p_2-1,1\right)\end{matrix}\right.\right]$$

$$
-\frac{a_3 s_8}{s_{10}} \sum_{p_1=0}^{\infty} \frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1\,!} \sum_{p_2=0}^{\infty} \left(\frac{s_9}{s_{10}}\right)^{p_2} \mathbf{M}_2^1 \left[\left(-s_8\right) \middle| \begin{matrix} \left(\frac{p_1}{2},1\right) \\ \left(\frac{p_1}{2},0\right), \left(-p_2-1,1\right) \end{matrix} \right]. \tag{33}
$$

$$
T(y,t) = 1 + \sum_{p_1=1}^{\infty} \frac{\left(-y\sqrt{s_7}\right)^{p_1}}{p_1\,!}\ \mathbf{M}_{p_2,1}^{\frac{p_1}{2}} \left(-s_8 t\right). \tag{34}
$$

Results and Discussions

This portion is fascinated to a comprehensive study of single walled and multi-walled CNTs with all pertinent parameters on velocity field and temperature distribution. Heat transfer analysis of single and multi-walls carbon nanotubes (SWCNTs and MWCNTs) for electrically conducting flow of Casson fluid is carried out. Single walled and multi-walled CNTs are suspended in methanol that is considered as a conventional base fluid. The governing partial differential equations of nanofluids have been modeled by Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. The comparison of analytical solutions for temperature distribution and velocity field has been established via both approaches i-e (AB) and (CF) fractional operators. The general analytical solutions are expressed in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}\left(T\right)$ and generalized **M**-function $\mathbf{M}_{q}^{p}\left(F\right)$ satisfying initial and boundary conditions. In order to have vivid rheological effects, the general analytical solutions in both cases (AB and CF fractional derivatives) are depicted for graphical illustrations and the comparison of three types of fluids (i) pure methanol (ii) methanol with single walls carbon nanotubes and (iii) methanol with multi-walls carbon nanotubes is portrayed via Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives. However, the major outcomes and consequences are expected as; the general solutions are investigated by introducing Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives and presented in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}\left(T\right)$ and generalized **M**-function $\mathbf{M}_{q}^{p}\left(F\right)$ which satisfy imposed conditions. Figs. 2(a) and 2(b) elucidate the effects of nanoparticles volume fraction on velocity field and temperature distribution. It is apparent from Fig. 2(a) that velocity fields investigated via Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) approach are increasing function with increasing nanoparticles volume fraction. On the contrary in Fig. 2(b), temperature field is decreasing by the variation of nanoparticles volume fraction through Caputo-Fabrizio (CF) approach. Meanwhile, Atangana-Baleanu (AB) approach has increasing behavior, both approaches have opposite trend as well. It is in accordance with the physical expectation that when the thermal conductivity is raised up then temperature of fluid is enlarged. Fig. 3 is depicted to elaborate the characteristics of transverse magnetic field which results the flow of velocity field in reciprocating influences. In brevity, the velocity field obtained by Atangana-Baleanu (AB) approach generates decelerations in fluid flow. Caputo-Fabrizio (CF) approach has reversal behavior in comparison with Atangana-Baleanu (AB) approach. It is also noted that the fluid flow is sequestrating and scattering over the whole vicinity of plate reciprocally. Fig. 4 is prepared for comparison of analytical solutions established by Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) approaches. The comparison of three types of nanofluids namely (i) pure methanol (ii) methanol with single walled carbon nanotubes and (iii) methanol with multi-walled carbon nanotubes is underlined by both fractional derivatives.

Conclusion

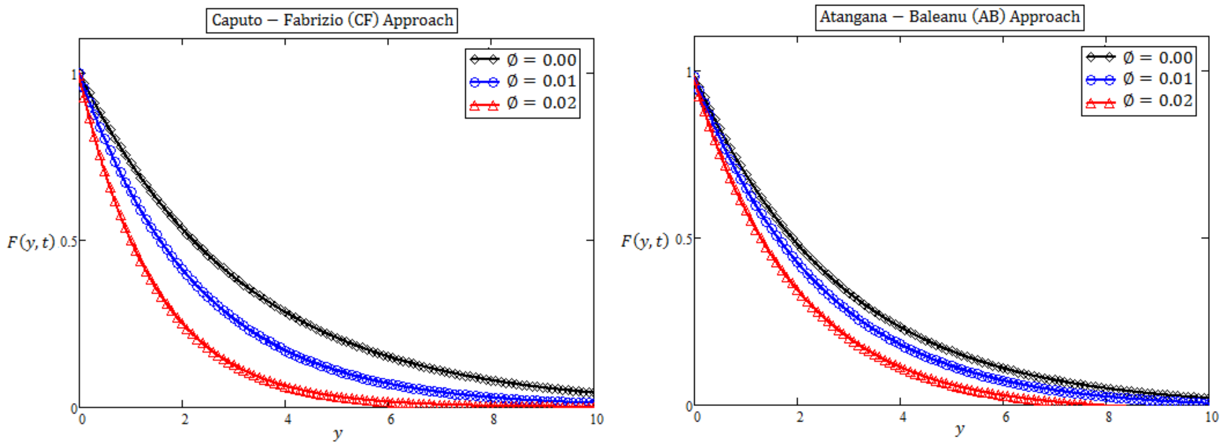
Applying newly defined fractional approaches (derivatives) namely Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) fractional derivatives on the governing partial differential equations of nanofluids, the comparison of analytical solutions for temperature distribution and velocity field has been investigated. The general analytical solutions have been obtained via Laplace transform and expressed in the layout of Mittage-Leffler function $\mathbf{M}_{\varepsilon,\delta}^{y}\left(T\right)$ and generalized **M**-function $\mathbf{M}_{q}^{p}\left(F\right)$ satisfying initial and boundary conditions. The major findings are listed below for the vivid rheological effects which are summarized below:

- The velocity field is investigated with Atangana-Baleanu (AB) approach has opposite trend of fluid flow than the velocity field with Caputo-Fabrizio (CF) approach. It is also noted that the velocity field with both approaches tends to nearer and nearer over the whole domain of plate.

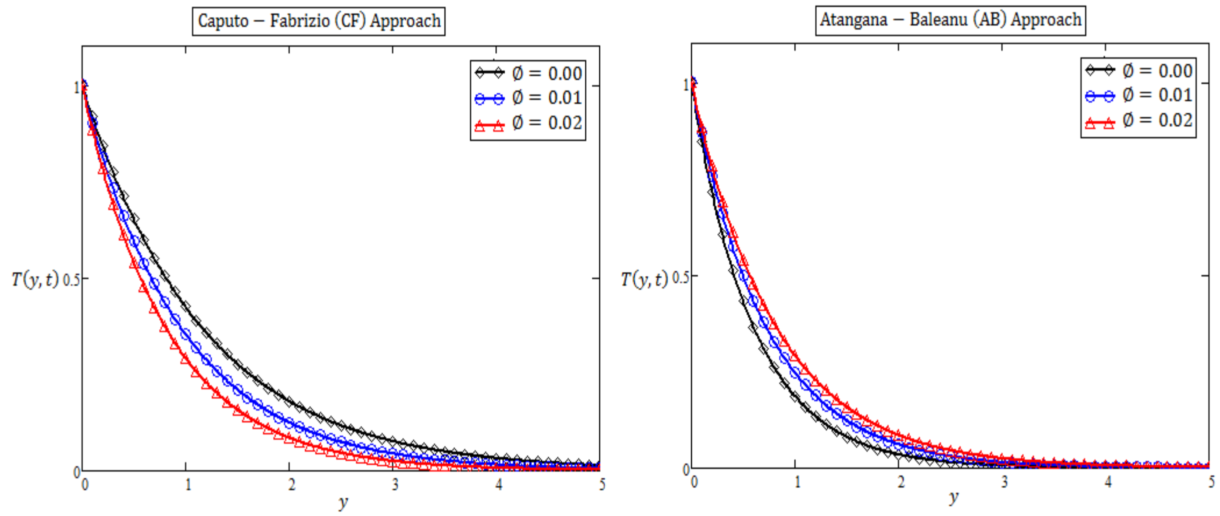
- The characteristics of transverse magnetic field results the flow of velocity field in reciprocating influences. Simply, the velocity field obtained by Atangana-Baleanu (AB) approach generates decelerations in fluid flow. On the contrary, Caputo-Fabrizio (CF) approach has reversal behavior in comparison with Atangana-Baleanu (AB) approach.
- The effects of nanoparticles volume fraction are investigated for temperature distribution as well as velocity field via Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) approaches are increasing and decreasing function respectively.
- The comparison of analytical solutions by Atangana-Baleanu (AB) and Caputo-Fabrizio (CF) approaches is also performed for three types of nanofluids. Here, pure methanol moves quicker in comparison with methanol-SWCNTs and methanol-MWCNTs. While methanol-MWCNTs moves more rapidly in comparison with pure methanol and methanol-SWCNTs. This leads to the phenomenon that multi-walled carbon nanotubes have slighter thermal conductivity as well as density than single walled carbon nanotubes.

Table. 1. Thermo-physical properties of Methanol and nanoparticles (CNTs) [51-52]

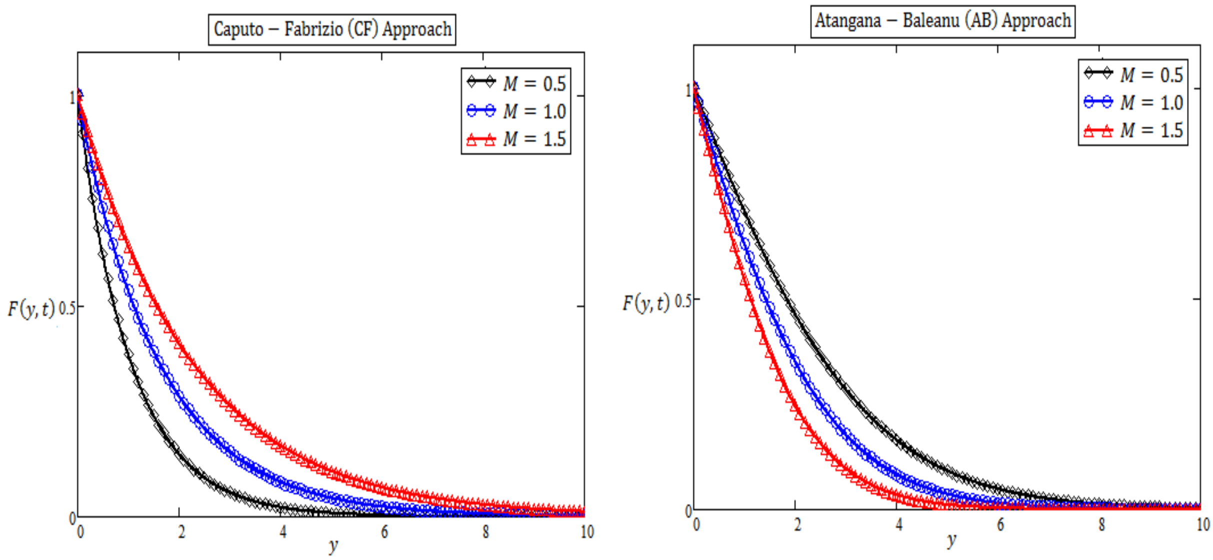
Base Fluid/Nanoparticles	ρ (Kg/m^3)	C_p (J/Kg K)	k (W/m.K)
Methanol	792	2545	0.2035
SWCNTs	2600	425	6600
SWCNTs	1600	796	3000



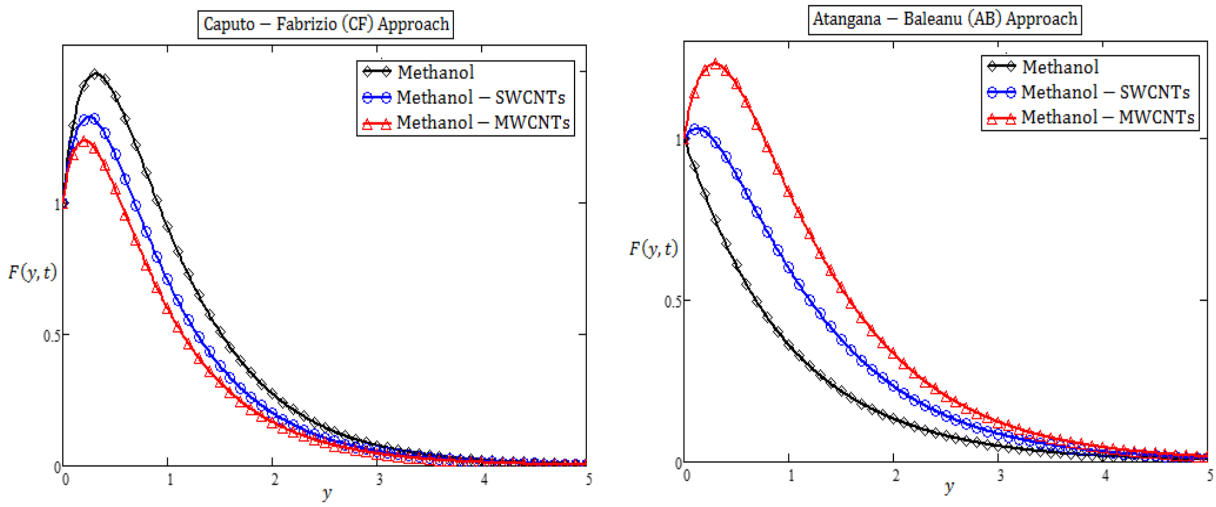
2(a). Profile of velocity field via Caputo-Fabrizio and Atangana-Baleanu fractional operators for nanoparticles volume fraction.



2(b). Profile of temperature distribution via Caputo-Fabrizio and Atangana-Baleanu fractional operators for nanoparticles volume fraction.



3. Profile of velocity field via Caputo-Fabrizio and Atangana-Baleanu fractional operators for magnetic field.



4. Comparison of velocity field via Caputo-Fabrizio and Atangana-Baleanu fractional operators for three types of models.

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