Uncertain fractional operator with application arising in the steady heat flow

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Abstract

In the recent years much efforts were made to propose simple and well-behaved fractional operators to inherit the classical properties from the first order derivative and overcome the singularity problem of the kernel appearing for the existing fractional derivatives. Therefore, we propose in this research an interesting approach to acquire the interval solution of fractional interval differential equations (FIDEs) under a new fractional operator, that does not have the above defect with uncertain parameters. In fact, this scheme is developed to achieve the interval solution of the uncertain steady heat flow based on the FIDEs. An example is experienced to illustrate our approach and validate it.

Keywords: Nonsingular kernel fractional derivative, Interval arithmetic, Interval-valued function, Steady heat flow.

1. Introduction

Although the foundation of the fractional differential equations (FDEs) was established since around three centuries ago [1–3], the last two decades witness a wide range of applications to the FDEs in various disciplinary to model the real-world systems [4–8]. The FDEs may represent such kind of phenomena more accurate than the traditional modeling. For example, the solute diffusion in contaminant flow in a groundwater aquifer as the does not follow Fickian law in general [9].

While these studies have been carried out, scientists used different definitions of fractional derivative and integral such as Grünwald-Letnikov, Riesz-Fischer, Liouville-Caputo, Riemann-Liouville, and modified Riemann-Liouville. But almost all of these derivatives have some kinds of flaws. For instance, the Riemann-Liouville fractional derivative of a constant is not zero, the Riemann-Liouville derivative and Liouville-Caputo derivative do not obey the Leibnitz rule and chain rule. The Riemann-Liouville derivative and Liouville-Caputo do not satisfy the known formula of the derivative of the quotient of two functions [10]. Besides, the singularity of the kernel in the integral of the formulas in the different types of fractional derivatives imposed a considerable complexity in modeling of the real-world systems. Due to the aforementioned circumstances, recently Caputo-Fabrizio fractional derivative was proposed in the literature to overcome this difficulty which was followed up by Losada and Nieto to propose some new operators for this derivative. However, the suggested fractional derivative was a type of generalization based on the Liouville-Caputo derivative and its relevant properties [11]. In this regards and very recently, Yang et al. [12] revealed a new fractional derivative without singular kernel, which is called Yang-Srivastava-Machado fractional derivative (for short YSM-fractional derivative), concerning the Riemann-Liouville derivative to study fractional equations arising in the steady heat flow.

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On the other hand, the interval arithmetic and interval differential equations (IDEs) have not been considered enough from the time that were introduced by Markov [16]. In fact, he was the pioneer in this field to propose the interval-valued function, interval derivatives and etc. However, in the recent years, the scientists found the applicability of this significant tool in the mathematical modeling with uncertain parameters. Therefore, a number of researches has been done in this regards to analyze the mathematical systems based on the interval parameters and study the existence and uniqueness of the interval solutions of the IDEs [17–21]. As a matter of fact, interval arithmetic is a branch of fuzzy sets that we deal with the intervals from the first step of modeling or numerical algorithm that can reduce the complexity and computational difficulties compared with fuzzy systems.

Considering the above circumstances and similar to the integer order differential equations, a limited number of researches has been devoted to the fractional interval differential equations (FIDEs), even a few studies have been done for fuzzy fractional calculus [22–24]. Because of the vast applications of interval and fuzzy arithmetic arising in current engineering problems [25], we were motivated to develop the generalized Riemann-Liouville derivative presented in [12] for the interval functions under uncertainty. Beside that, we model the steady heat flow problem based on the interval arithmetic. This new and novel approach can open a new window to model the engineering problems based on this new fractional derivative under interval uncertainty. To the best of authors knowledge, there is not any report in the literature to get the interval/fuzzy solution of FIDEs or fuzzy fractional DEs with this new fractional derivative.

The paper is organized as follows: in Section 2, some important concepts of interval arithmetic are revisited. Besides, the definition of the non-interval concept of the new fractional derivative is also recalled. Section 3 discusses about the solution of the FIDEs based on the interval fractional derivative. In this section, a number of important theorems are proved to support the conditions which under those, the solution of the interval equation of the steady heat flow problem is exploited. Afterwards, a type of interval model of the steady heat flow is solved using the proposed approach to get the interval solutions and demonstrate the validity of our present strategy. Finally, some conclusion remarks are drawn in the last section.

2. Preliminaries and notation

It is important to highlight the basic assumptions and definitions in this research before we present the main results. In this regards, we first revisit the necessary discussions of the interval arithmetic presented in [20, 26] and fractional calculus theory [1, 2].

2.1. Interval arithmetic

Let suppose that $\mathcal{I}_{\mathcal{F}}$ represents the family of all nonempty, compact and convex intervals of the real line $\mathbb{R}$. Suppose that $\mathrm{G}, \mathrm{H} \in \mathcal{I}_{\mathcal{F}}$. If there exists an interval $\mathrm{J} \in \mathcal{I}_{\mathcal{F}}$ such that $\mathrm{G} = \mathrm{H} + \mathrm{J}$, then we state J the *Hukuhara difference* (H-difference for short) of G and H. We show the interval J by $\mathrm{G} \ominus \mathrm{H}$. Consider the fact that $\mathrm{G} \ominus \mathrm{H} \neq \mathrm{G} + (-1)\mathrm{H}$. Regarding to the above explanation, the H-difference is unique, but it does not always exist. A generalization of the this type of difference is stated in [20] to overwhelm this defect.

Definition 2.1. *The generalized Hukuhara difference of two interval numbers $u_1, u_2 \in \mathcal{I}_{\mathcal{F}}$ (gH-difference for sake of simplicity) is defined as follows*

$$u_1 \ominus_g u_2 = u_3 \Leftrightarrow \begin{cases} (i)\ u_1 = u_2 + u_3, \\ \qquad or \\ (ii)\ u_2 = u_1 + (-1)u_3, \end{cases} \tag{2.1}$$

in which $u_3 \in \mathcal{I}_{\mathcal{F}}$.

The most significant definition of the interval derivative, based on the fuzzy differentiability notion presented in [27], is stated as:

Definition 2.2. *Assume that $\mathcal{U} : (a, b) \to \mathcal{I}_{\mathcal{F}}$ and $\tau \in (a, b)$. We call that $\mathcal{U}$ is strongly generalized (Hukuhara) differentiable at τ, if there exists an element $\mathcal{U}'(\tau) \in \mathcal{I}_{\mathcal{F}}$, such that $\mathcal{U}'(\tau)$ satisfies in one of the following cases:*

(I) for all $h > 0$ sufficiently small, $\exists \mathcal{U}(\tau + h) \ominus \mathcal{U}(\tau)$, $\exists \mathcal{U}(\tau) \ominus \mathcal{U}(\tau - h)$ and

$$\lim_{h \searrow 0} \frac{\mathcal{U}(\tau + h) \ominus \mathcal{U}(\tau)}{h} = \lim_{h \searrow 0} \frac{\mathcal{U}(\tau) \ominus \mathcal{U}(\tau - h)}{h}, = \mathcal{U}'(\tau)$$

(II) for all $h > 0$ sufficiently small, $\exists \mathcal{U}(\tau) \ominus \mathcal{U}(\tau + h)$, $\exists \mathcal{U}(\tau - h) \ominus \mathcal{U}(\tau)$ and

$$\lim_{h \searrow 0} \frac{\mathcal{U}(\tau) \ominus \mathcal{U}(\tau + h)}{-h} = \lim_{h \searrow 0} \frac{\mathcal{U}(\tau - h) \ominus \mathcal{U}(\tau)}{-h}, = \mathcal{U}'(\tau)$$

If $\mathcal{U}'(\tau) \in \mathcal{I}_{\mathcal{F}}$ satisfying Definition 2.2 exists, we say that $\mathcal{U}$ is generalized Hukuhara (gH)-differentiable at τ. Also, we say that $\mathcal{U}$ is [(I)-gH]-differentiable at τ, if τ satisfies in Definition (2.2)-(I), then we have $\mathcal{U}'_{gH}(\tau) = [\underline{\mathcal{U}}'(\tau), \overline{\mathcal{U}}'(\tau)]$, similarly, τ is [(II)-gH]-differentiable at τ, if $\mathcal{U}$ satisfies in Definition (2.2)-(II), then we have $\mathcal{U}'_{gH}(t) = [\overline{\mathcal{U}}'(\tau), \underline{\mathcal{U}}'(\tau)]$.

Remark 2.1. *The interval definitions and properties were recalled from the fuzzy research reports. It is worth noting here that all the fuzzy definitions which were reviewed here, hold for the interval cases as they are the special cases of the fuzzy notion.*

Proposition 2.1. *We call that an interval-valued function $f : [a, b] \to \mathcal{I}_{\mathcal{F}}$ is w-increasing (w-decreasing) on $[a, b]$ if the real function $t \to w_F(t) := w(F(t))$ is increasing (decreasing) on $[a, b]$. If F is w-increasing or w-decreasing on $[a, b]$, then we call that F is w-monotone on $[a, b]$ (see, [17]).*

2.2. Yang-Srivastava-Machado fractional derivative

In this part, we recall the definition of the YSM-fractional derivative [12].

Definition 2.3. *The YSM-fractional derivative is defined as:*

$$^{YSM}\mathfrak{D}_{a^+}^{(\alpha)} P(x) = \frac{\mathcal{K}(\alpha)}{1 - \alpha} \frac{d}{dx} \int_a^x \exp\left[-\frac{\alpha}{1 - \alpha}(x - \tau)\right] P(\tau)\, d\tau \tag{2.2}$$

where $a \leq x$ and $\alpha(0 < v < 1)$ is a real number, and $\mathcal{K}(\alpha)$ is the normalization function that depends on α such that $\mathcal{K}(0) = \mathcal{K}(1) = 1$.

If we assume for a special order of α in which $0 < \alpha < 1$ and $\mathcal{K} = 1$, then the Eq. (2.2) is rewritten as follows:

$$^{YSM}\mathfrak{D}_{a^+}^{(\alpha)} P(x) = \frac{1}{1 - \alpha} \int_a^x \exp\left[-\frac{\alpha}{1 - \alpha}(x - \tau)\right] P(\tau) d\tau, \tag{2.3}$$

and

$$P(x) = (1 - \alpha)\Psi(x) + \alpha \int_0^x \Psi(x) dx, \quad x > 0, \quad 0 < \alpha < 1. \tag{2.4}$$

where $^{YSM}D_{a^+}^{(\alpha)} P(x) = \Psi(x)$ and $\mathcal{L}[\Psi(x)] = \Psi(x)$. in which $\mathcal{L}$ stands for the Laplace transform operator.

3. Main results

In what discuss after, we aim to propose an uncertain version of the YSM-fractional derivative to solve the steady heat flow with interval uncertain parameters. Th outcomes will be employed in the next part considering solutions of the developed FIDE under YSM-fractional gH-differentiability.

Definition 3.1. *Let suppose that $\mathcal{AC}([a, b], \mathcal{I}_{\mathcal{F}})$ indicates the space of absolute continuous interval functions on $[a, b]$ and $f \in \mathcal{AC}([a, b], \mathcal{I}_{\mathcal{F}})$, we say that f is YSM-differentiable at point $x \in (a, b)$ of order α for $\alpha \in (0, 1)$, if there exists $^{YSM}f^{(\alpha)}(x) \in \mathcal{I}_{\mathcal{F}}$ such that*

$$^{YSM}f^{(\alpha)}(x) = \frac{\mathcal{K}(\alpha)}{1 - \alpha} \frac{d}{dx} \int_a^x \exp\left[-\frac{\alpha}{1 - \alpha}(x - \tau)\right] f(\tau) d\tau \tag{3.5}$$

in which similar to the crisp case, $\mathcal{K}(\alpha)$ is the normalization function that depends on α.

Theorem 3.1. *Let $f(x) = [f_1(x), f_2(x)] \in \mathcal{AC}([a,b], \mathcal{I}_{\mathcal{F}})$ then we have:*

$$^{YSM}f^{(\alpha)}(x) \supseteq \left[\min\{^{YSM}f_1^{(\alpha)}(x), {}^{YSM}f_2^{(\alpha)}(x)\}, \max\{^{YSM}f_1^{(\alpha)}(x), {}^{YSM}f_2^{(\alpha)}(x)\}\right], \tag{3.6}$$

for a.e. $x \in (a,b)$. Also, if f is w-increasing, then $^{YSM}f^{(\alpha)}(x) = [^{YSM}f_1^{(\alpha)}(x), {}^{YSM}f_2^{(\alpha)}(x)]$ and also if f is w-decreasing then $^{YSM}f^{(\alpha)}(x) = [^{YSM}f_2^{(\alpha)}(x), {}^{YSM}f_1^{(\alpha)}(x)]$ for a.e. $x \in (a,b)$.

Proof. As $f \in \mathcal{AC}([a,b], \mathcal{I}_{\mathcal{F}})$, then f_1, f_2 are differentiable and therefore $^{YSM}f_1^{(\alpha)}(x)$ and $^{YSM}f_2^{(\alpha)}(x)$ exist. Hence, we have:

$$\begin{aligned}
^{YSM}f^{(\alpha)}(x) = {} & \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] f(\tau)d\tau \\
& \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\int_a^x [\min\{f_1(\tau), f_2(\tau)\}, \max\{f_1(\tau), f_2(\tau)\}] \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau \\
& \supseteq \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}[\min\{\int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] f_1(\tau)d\tau, \int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] f_2(\tau)d\tau\}, \\
& \max\{\int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] f_1(\tau)d\tau, \int_a^x \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] f_2(\tau)d\tau\}].
\end{aligned}$$

Indeed,

$$\begin{aligned}
^{YSM}f^{(\alpha)}(x) &= \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\int_a^x \min\{f_1(\tau), f_2(\tau)\} \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau \\
&\leqslant \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\min\{\int_a^x f_1(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau, \int_a^x f_1(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau\} \\
&\leqslant \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\max\{\int_a^x f_1(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau, \int_a^x f_1(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau\} \\
&\leqslant \frac{\mathcal{K}(\alpha)}{1-\alpha}\frac{d}{dx}\int_a^x \max\{f_1(\tau), f_2(\tau)\} \exp\left[-\frac{\alpha}{1-\alpha}(x-\tau)\right] d\tau,
\end{aligned}$$

that the proof of the theorem is completed. □

Remark 3.1. *Indeed, it is easy to verify that*

$$\lim_{\alpha \to 1} {}^{YSM}f^{(\alpha)}(x) = f'(x),$$

$$\lim_{\alpha \to 0} {}^{YSM}f^{(\alpha)}(x) = f(x).$$

Theorem 3.2. *Let assume that $f(x) = [f_1(x), f_2(x)] \in \mathcal{AC}([a,b], \mathcal{I}_{\mathcal{F}})$, then we have:*

$$\mathcal{L}\{^{YSM}f^{(\alpha)}(x)\} = \frac{\mathcal{K}(\alpha)}{\alpha(1-s)+s} f(s)$$

Proof. The proof is straightforward. □

3.1. YSM-fractional model of steady heat flow under interval uncertainty

Similar to results proposed in [12], we will develop the YSM-fractional FIDEs for steady heat flow under interval fractional differentiability. Applying this new approach, we give solutions to interval initial-value problems for fractional-order IDEs of the original problem.

Let consider the following fractional interval Fourier law in 1-D station as:

$$\mathcal{T}^{YSM}f^{(\alpha)}(x) = -\mathcal{H}(x), \tag{3.7}$$

in which $\mathcal{T}$ stands for the thermal conductivity of the material and $\mathcal{H}(x)$ indicates the heat flux density and we assume here that both of them are the interval parameters under uncertainty.

If we consider $\mathcal{H}(x) = q$ where q the uncertain heat flow of the material, then, using Laplace transform operator and taking into account the Eq. (2.4), we have:

$$\frac{\mathcal{K}(\alpha)}{\alpha(1-s)+s} f(s) = \frac{q}{\mathcal{T}},$$

then,

$$f(s) = \frac{-q[\alpha + (1-s)]}{\mathcal{T}\mathcal{K}(\alpha)s},$$

using inverse Laplace transform, we have:

$$f(x) = -\lambda \left[\frac{q\alpha x}{\mathcal{T}\mathcal{K}(\alpha)} + \frac{q(1-\alpha)}{\mathcal{T}\mathcal{K}(\alpha)} \right],$$

in which λ is a constant parameter that depends on the initial value of $f(x)$.
For instance, set $q = [q_1, q_2] = [1, 2]$, then we have:

$$f(x) = -\lambda \left(\frac{[1,2](\alpha x + (1-\alpha)}{\mathcal{T}\mathcal{K}(\alpha)} \right).$$

If we assume in the above statement that $\lambda = -1$ and $\mathcal{K}(\alpha) = \mathcal{T} = 1$, then we have:

$$f(x) = [1,2](\alpha x + (1-\alpha)). \tag{3.8}$$

Fig. 1 demonstrates the validity of the proposed interval fractional derivative for the problem (3.7) with the assumed values of the parameters based on the different the fractional-orders over $[0, 5]$. It is obvious from the figure that these solutions may have a diminishing length of their support, which is a critical feature to reflect the expected behavior of interval solutions of FIDEs based on interval YSM-fractional derivative.

4. Conclusion

In this research work, we proposed a plan to obtain an explicit interval solution for a type of FIDEs based on a new fractional derivative suggested in [12] under interval differentiability. In fact, the YSM-fractional derivative was developed for interval uncertainty to model the uncertain steady heat flow. This matter helps us to find the solutions of the modeled real-world system explicitly under the assumption of the interval YSM-fractional derivative.
For future plan, we aim to extend this new approach for the other kinds of IDEs such as random IDEs, functional IDEs based on the discussed fractional derivative.

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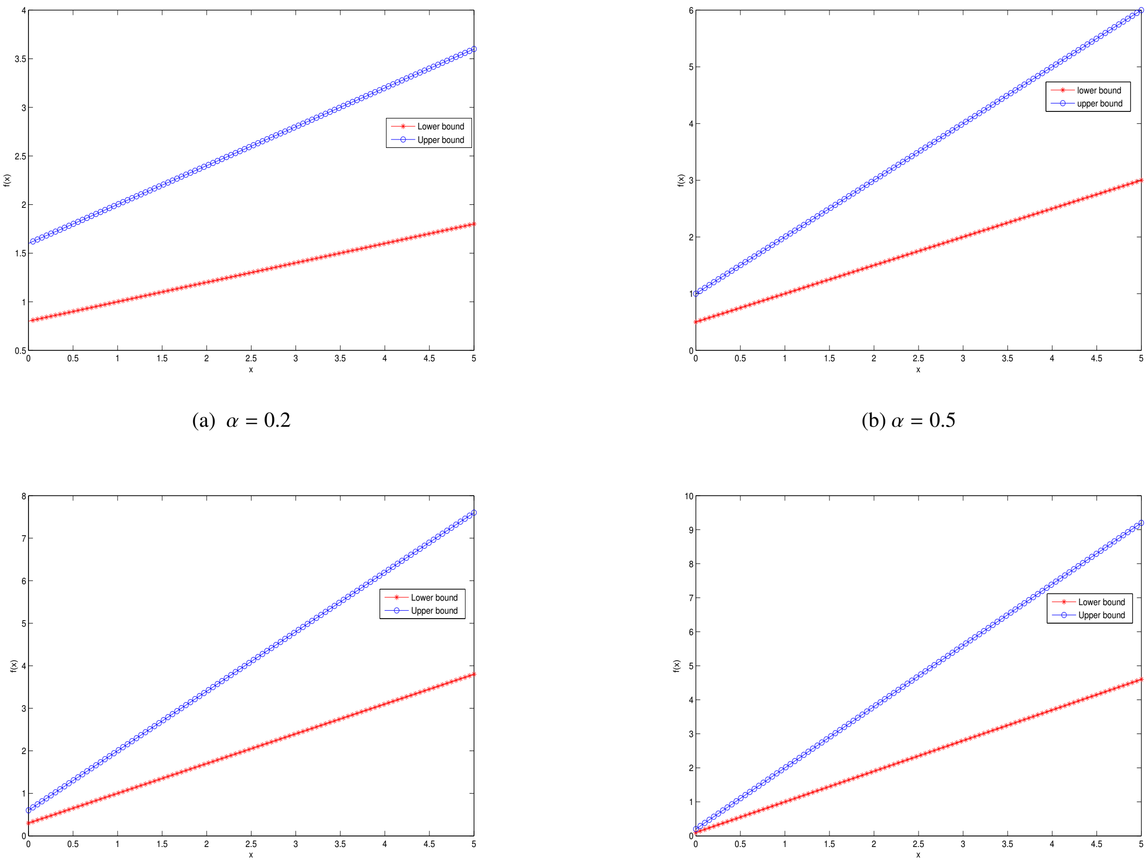

(a) $\alpha = 0.2$ (b) $\alpha = 0.5$

(c) $\alpha = 0.7$ (d) $\alpha = 0.9$

Figure 1: Interval solution with different fractional-order, α, for Eq. 3.8 over $x \in [0, 5]$.

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