THE STUDY OF HEAT TRANSFER PHENOMENA BY USING MODIFIED HOMOTOPY PERTURBATION METHOD COUPLED BY LAPLACE TRANSFORM

Uriel FILOBELLO-NINO[1], Hector VAZQUEZ-LEAL[1]*, Agustin Leobardo HERRERA-MAY,[2,3], Roberto Carlos AMBROSIO-LAZARO[3], Victor Manuel JIMENEZ-FERNANDEZ[1], Mario Alberto SANDOVAL-HERNANDEZ[4], Oscar ALVAREZ-GASCA[1], Beatriz Elena PALMA-GRAYEB[1].

[*1]Facultad de Instrumentación Electrónica, Universidad Veracruzana, Circuito Gonzalo Aguirre Beltrán S/N, Xalapa, Veracruz, 91000, México

[2] Micro and Nanotechnology Research Center, Universidad Veracruzana, Calzada Ruiz Cortínes 455, 94294 Boca del Río, Veracruz, México.

[3]Maestría en Ingeniería Aplicada, Facultad de Ingeniería de la Construcción y el Hábitat, Universidad Veracruzana, Calzada Ruíz Cortines 455, 94294, Boca del Río, Veracruz, Mexico.

[4]National Institute of Astrophysics, Optics and Electronics, Luis Enrique Erro # 1, Santa María Tonatzintla, 72840, Puebla, México.

[*]Corresponding author; E-mail: hvazquez@uv.mx

In this paper, we present Modified Homotopy Perturbation Method Coupled by Laplace Transform (MHPMLT) to solve nonlinear problems. As case study MHPMLT is employed in order to obtain an approximate solution for the nonlinear differential equation that describes the steady state of a heat one-dimensional flow. The comparison between approximate and exact solutions shows the practical potentiality of the method.

Key words*: Homotopy Perturbation Method; Laplace Transform; Heat conduction*

1. Introduction

The heat transfer laws are of paramount importance in the design and operation of equipment in many industrial applications as well as in pure sciences. Therefore, it is important to search for accurate analytical approximate solutions for the equations describing these phenomena. However, it is well known that nonlinear differential equations that describe them are difficult to solve [1-3].

Laplace Transform (L.T.) plays a relevant role in mathematics because its methods allow to solve many problems in science and engineering [4]. It is well known that Laplace Transform is a powerful tool useful for solving linear ordinary differential equations with constant coefficients and initial conditions and to solve some cases of differential equations with variable coefficients and partial differential equations [4]. The contribution of L.T. to nonlinear ordinary differential equations has required its combination with other techniques. Thus, [1] reported a combination of Homotopy Perturbation (HPM) and L.T. methods (LT-HPM) in order to obtain highly accurate approximate solutions for these equations. At this time, it is clafiried that the coupling of LT and HPM is known by another name in the literature: A modified Homotopy Perturbation Method coupled by Laplace Transform (MHPMLT) [5,6,7]. As a matter of fact, [5] named this modification as He-Laplace

Method for simplicity. Thus, from here on, we agree to call the proposed method MHPMLT. This work proposes MHPMLT method in the search for approximate solutions for the nonlinear ordinary differential equation with Dirichlet boundary conditions defined on a finite interval [8] which describes the steady state one-dimensional heat conduction in a slab with temperature-dependent thermal conductivity [9]. The case of equations with boundary conditions on infinite intervals has been reported for some authors [10,11] and correspond often to problems defined on semi-infinite ranges. However, the methods for solving these problems are different from those that will be presented in this study [8]. Nonlinear problems frequently arise in science and engineering, whereby, it is very important to search on differential equations that describe them. In recent years, there have been proposed several methods focused to find approximate solutions to nonlinear differential equations; such as those based on: variational approaches [12], tanh method [13], exp-function [14], Adomian's decomposition method (ADM) [15], parameter expansion [16], homotopy analysis method (HAM) [2,3], perturbation method [17], and homotopy perturbation method (HPM) [1,5-11,18-23], since the main solution process of this article is HPM, next we briefly mention some of the last developments of this method; such as the coupling of HPM and Frobenius method [20], multiple scales HPM method [21], Parametrized HPM [22], Nonlinearities Distribution Homotopy Perturbation Method used to find the solution of Troesch Problem [23]; among many others.

The paper is organized as follows. In Section 2, we introduce the basic idea of standard HPM method. For Section 3 we introduce the necessary aspects of MHPMLT. Additionally in Section 4 the basic equations for the heat transfer problem mentioned are derived. Section 5 presents the application of the proposed method, in the search for an approximate solution for the nonlinear ordinary differential equation, which describes the steady state one-dimensional conduction of heat in a slab with temperature-dependent thermal conductivity. Besides, a discussion on the results is presented in Section 6. Finally, a brief conclusion is given in Section 7.

2. Standard HPM

The standard homotopy perturbation method (HPM) was proposed by Ji Huan He, it was introduced like a powerful tool to approach various kinds of nonlinear problems. The Homotopy Perturbation Method (HPM) is considered as a combination of the classical perturbation technique and the homotopy (whose origin is in the topology), but not restricted to small parameters as occur with traditional perturbation methods. For example, HPM method requires neither small parameter nor linearization, but only few iterations to obtain highly accurate solutions [19,24].

To figure out how HPM works, consider a general nonlinear differential equation in the form

$$A(u) - f(r) = 0,\ r \in \Omega, \tag{1}$$

with the following boundary conditions

$$B(u, \partial u / \partial n) = 0,\ \ r \in \Omega, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ a known analytical function and Γ is the domain boundary for Ω. A can be divided into two operators L and N, where L is linear and N nonlinear; so that (1) can be rewritten as

$$L(u)+N(u)-f(r)=0\,. \tag{3}$$

Generally, a homotopy can be constructed as [19,24]

$$H(U,p)=(1-p)[L(U)-L(u_0)]+p[L(U)+N(U)-f(r)]=0\,,\quad p\in[0,1],\; r\in\Omega\,. \tag{4}$$

or

$$H(U,p)=L(U)-L(u_0)+p[L(u_0)+N(U)-f(r)]=0\,,\qquad p\in[0,1]\,, \tag{5}$$

where p is a homotopy parameter, whose values are within range of 0 and 1, and u_0 is the first approximation for the solution of (3) that satisfies the boundary conditions.

Assuming that solution for (4) or (5) can be written as a power series of p as

$$U=v_0+v_1p+v_2p^2+... \tag{6}$$

Substituting (6) into (5) and equating identical powers of p terms there can be found values for the sequence $v_0, v_1, v_2, \ldots$

When $p\to 1$ yields to the approximate solution for (1) in the form

$$U=v_0+v_1+v_2+v_3... \tag{7}$$

3. Basic Idea of Modified Homotopy Perturbation Method Coupled by Laplace Transform

The objective of this section is showing how MHPMLT can be employed to find analytical approximate solutions for ordinary differential equations (ODES) as (3).

For this purpose, MHPMLT follows the same steps of standard HPM until (5), next we apply Laplace transform on both sides of homotopy equation (5), to obtain

$$\Im\left\{L(U)-L(u_0)+p\left[L(u_0)+N(U)-f(r)\right]\right\}=0 \tag{8}$$

using the differential property of L.T, we have [4]

$$s^n\Im\{U\}-s^{n-1}U(0)-s^{n-2}U'(0)-...-U^{(n-1)}(0)=\Im\left\{L(u_0)-pL(u_0)+p\left[-N(U)+f(r)\right]\right\} \tag{9}$$

or

$$\Im(U)=\left(\frac{1}{s^n}\right)\left\{s^{n-1}U(0)+s^{n-2}U'(0)+..+U^{(n-1)}(0)\right\}+\left(\frac{1}{s^n}\right)\Im\left\{L(u_0)-pL(u_0)+p\left[-N(U)+f(r)\right]\right\} \tag{10}$$

applying inverse Laplace transform to both sides of (10), we obtain

$$U = \Im^{-1}\left\{\begin{array}{l}\left(\dfrac{1}{s^n}\right)\left\{s^{n-1}U(0)+s^{n-2}U'(0)+..+U^{(n-1)}(0)\right\}+\\ \left(\dfrac{1}{s^n}\right)\Im\left\{L(u_0)-pL(u_0)+p\left[-N(U)+f(r)\right]\right\}\end{array}\right\} \tag{11}$$

Assuming that solutions of (3) can be expressed as a power series of p

$$U=\sum_{n=0}^{\infty}p^n v_n \tag{12}$$

then substituting (12) into (11), we get

$$\sum_{n=0}^{\infty}p^n v_n = \Im^{-1}\left\{\begin{array}{l}\left(\dfrac{1}{s^n}\right)\left\{s^{n-1}U(0)+s^{n-2}U'(0)+..+U^{(n-1)}(0)\right\}\\ +\Im\left\{L(u_0)-pL(u_0)+p\left[-N(\sum\limits_{n=0}^{\infty}p^n v_n)+f(r)\right]\right\}\end{array}\right\}, \tag{13}$$

comparing coefficients of p, with the same power leads to

$$\begin{aligned}
p^0 &: v_0 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\left(s^{n-1}U(0)+s^{n-2}U'(0)+..+U^{(n-1)}(0)\right)+\Im\left\{L(u_0)\right\}\right\},\\
p^1 &: v_1 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\left(\Im\left\{-N(v_0)-L(u_0)+f(r)\right\}\right)\right\},\\
p^2 &: v_2 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\left\{-N(v_0,v_1)\right\}\right\},\\
p^3 &: v_3 = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\left\{-N(v_0,v_1,v_2)\right\}\right\},\\
&\dots\\
p^j &: v_j = \Im^{-1}\left\{\left(\frac{1}{s^n}\right)\Im\left\{-N(v_0,v_1,v_2,\dots,v_j)\right\}\right\},\\
&\dots
\end{aligned} \tag{14}$$

Assuming that the initial approximation has the form: $U(0)=u_0=\alpha_0,\ U'(0)=\alpha_1,..,U^{n-1}(0)=\alpha_{n-1}$; then the approximate solution may be obtained as follows

$$u=\lim_{p\to 1}U=v_0+v_1+v_2+\dots \tag{15}$$

4. Governing equations.

The goal of this work is the searching for an approximate solution for the nonlinear problem, which describes the steady state one-dimensional conduction of heat in a slab with thermal conductivity linearly dependent on the temperature (see Figure 1).

The transferred energy caused by the temperature difference between two adjacent parts of a body is called heat conduction [25]. Let $u(x,y,z,t)$ be the temperature of the above mentioned slab at a point (x,y,z) at time t, and K, σ and μ the thermal conductivity, specific heat, and density of the solid respectively; then it is verified that the temperature obeys the following partial differential equation [26].

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right)=\sigma\mu\frac{\partial u}{\partial t} \tag{16}$$

which is known as the heat conduction equation.

In the cases where K is a constant, the above equation reduces to the following known linear partial differential equation.

$$K\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\sigma\mu\frac{\partial u}{\partial t} \tag{17}$$

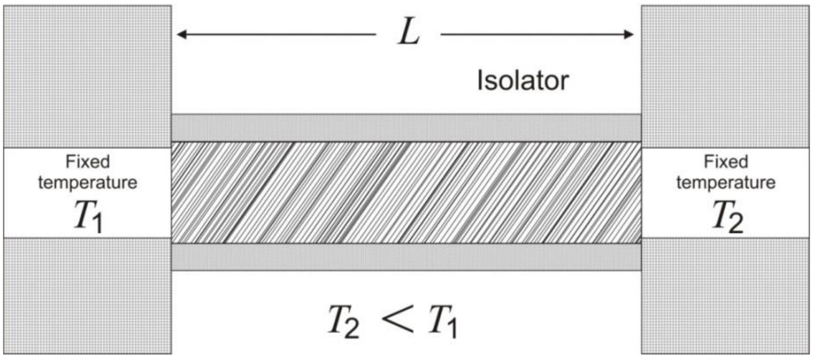

Figure 1 shows a one-dimensional conduction of heat through an insulated slab.

A substance with a high thermal conductivity is a good heat conductor; on the contrary, one with a small thermal conductivity is a poor conductor of heat, or equivalently a good thermal insulator. The K value depends on the temperature, increasing slightly when it increases, but can be considered almost constant throughout a substance if the temperature difference between the parts is not too large [25].

In the case of heat flow under steady conditions, the temperature does not depend on time t, such that $\partial u/\partial t=0$. Thus (16) becomes in

$$\frac{\partial}{\partial x}\left(K\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial u}{\partial z}\right)=0 \tag{18}$$

while (17) becomes in the Laplace equation for u

$$\nabla^2 u=0 \tag{19}$$

Solutions to the linear heat conduction equations for constant thermal conductivity (17) and (19) are studied in detailed, for instance in [25,26]. Nonetheless unlike the above, in general K is dependent on temperature, and in this case (16) and (18), are nonlinear.

This article considers the case of steady conditions for the one dimensional conduction of heat in a slab of thickness L, assuming a temperature dependent thermal conductivity K [9]. Supposing that the temperatures of the two opposite faces of the slab are uniformly maintained at T_1 and T_2, where $T_2 < T_1$; then, the governing equation is obtained as a one dimensional case of (18), so that

$$\frac{d}{dx}\left(K\frac{du}{dx}\right)=0 \tag{20}$$

subject to the following boundary conditions

$$u(0) = T_1, \quad u(L) = T_2. \tag{21}$$

For the sake of simplicity, we assume that the thermal conductivity varies linearly with temperature, thus [9] (see discussion).

$$K = K_2\left[1 + \beta\left(u - T_2\right)\right] \tag{22}$$

where K_2 is the thermal conductivity at temperature T_2 and the constant β is defined below.

In order to employ MHPMLT to obtain a handy accurate analytical approximate solution for the heat problem above described, we rewrite (20) as follows. After performing the indicated derivative in (20), we get

$$K\frac{d^2u}{dx^2} + \frac{dK}{dx}\frac{du}{dx} = 0 \tag{23}$$

Next, it is suggested the introduction of the following dimensionless quantities [9].

$$y = \frac{u - T_2}{T_1 - T_2}, z = \frac{x}{L}, \varepsilon = \beta(T_1 - T_2) = \frac{K_1 - K_2}{K_2}, \tag{24}$$

note that the last equality of (24) defines β.

Employing the chain rule, it is possible to deduce that

$$\frac{d}{dx} = \frac{1}{L}\frac{d}{dz}, \frac{d^2}{dx^2} = \frac{1}{L^2}\frac{d^2}{dz^2} \tag{25}$$

On the other hand, the first equation of (24) can be written as

$$u = T_2 + y(T_1 - T_2) \tag{26}$$

Thus, substituting (26) into (22), yields in

$$K = K_2\left[1 + \varepsilon y\right] \tag{27}$$

After substituting (25)-(27) into (23), it is obtained

$$\frac{d^2y}{dz^2} + \varepsilon y\frac{d^2y}{dz^2} + \varepsilon\left(\frac{dy}{dz}\right)^2 = 0 \tag{28}$$

where boundary conditions (21) adopt the simpler form

$$y(0) = 1, \quad y(1) = 0. \tag{29}$$

5. Case Study

The objective of this section is employ MHPMLT, to find an analytical approximate solution for the nonlinear problem given by (28) and (29).

We will see that it is possible to find a handy solution by applying MHPMLT method.

Identifying terms:

$$L(y) = y''(z) \tag{30}$$

$$N(y) = \varepsilon(y''(z)y(z) + y'^2(z)), \tag{31}$$

where prime denotes differentiation respect to z.
In accordance with (4), we propose

$$(1-p)(y''-y''_0) + p\left[y'' + \varepsilon yy'' + \varepsilon y'^2\right] = 0 \tag{32}$$

or

$$y'' = y''_0 + p\left[-y''_0 - \varepsilon yy'' - \varepsilon y'^2\right] \tag{33}$$

Applying Laplace transform we get,

$$s^2Y(s) - A - s = \Im\left(y''_0 + p\left(-y''_0 - \varepsilon yy'' - \varepsilon y'^2\right)\right) \tag{34}$$

where we have defined $Y(s) = \Im(y(z))$, $A = y'(0)$ with initial condition $y(0) = 1$.
Solving for $Y(s)$ and applying Laplace inverse transform $\Im^{-1}$

$$y(z) = \Im^{-1}\left\{\frac{1}{s} + \frac{A}{s^2} + \frac{1}{s^2}\Im\left(y''_0 + p\left(-y''_0 - \varepsilon yy'' - \varepsilon y'^2\right)\right)\right\} \tag{35}$$

Next, we assume a series solution for $y(z)$, in the form

$$y(z) = \sum_{n=0}^{\infty} p^n v_n(z) \tag{36}$$

where

$$v_0(z) = 1 + Az \tag{37}$$

is chosen as the first approximation for the solution of (28) that satisfies the conditions $y(0) = 1$, $y'(0) = A$.

Substituting (36) and (37) into (35), we get

$$\begin{aligned}\sum_{n=0}^{\infty} p^n v_n = \Im^{-1}\left\{\frac{1}{s} + \frac{A}{s^2} + \frac{1}{s^2}\Im\left(y''_0 + p\left(-y''_0 - \varepsilon\left(v_0 + pv_1 + p^2v_2 + ..\right)\left(v''_0 + pv''_1 + p^2v''_2 + ..\right)\right)\right)\right\} \\ -\varepsilon\Im^{-1}\left\{\frac{1}{s^2}\Im\left\{\left(v'_0 + pv'_1 + p^2v'_2 + ..\right)^2\right\}\right\}.\end{aligned} \tag{38}$$

On comparing the coefficients of like powers of p we have

$$p^0 : v_0(z) = \Im^{-1}\left\{\frac{1}{s} + \frac{A}{s^2}\right\}, \tag{39}$$

$$p^1 : v_1(z) = \Im^{-1}\left\{\left(\frac{-\varepsilon}{s^2}\right)\Im\{v_0v''_0 + v'^2_0\}\right\}, \tag{40}$$

$$p^2: \nu_2(z) = \Im^{-1}\left\{\left(\frac{-\varepsilon}{s^2}\right)\Im\{\nu_0\nu_1'' + \nu_1\nu_0'' + 2\nu_0'\nu_1'\}\right\}, \tag{41}$$

K

After solving the above Laplace transforms for $\nu_0(z),\ \nu_1(z),\ \nu_2(z),..$ we obtain

$$p^0: \nu_0(z) = 1 + Az, \tag{42}$$

$$p^1: \nu_1(z) = \frac{-\varepsilon A^2 z^2}{2}, \tag{43}$$

$$p^2: \nu_2(z) = \varepsilon^2\left[\frac{A^2z^2}{2} + \frac{A^3z^3}{2}\right]. \tag{44}$$

K

and so on.

By substituting (42)-(44) into (15) and evaluating the limit $p \to 1$, results in a handy second order approximation.

$$y(z) = 1 + Az + \frac{\varepsilon A^2}{2}\left(-1+\varepsilon\right)z^2 + \frac{\varepsilon^2}{2}A^3z^3. \tag{45}$$

In order to calculate the value of A, we require that (45) satisfies the boundary condition $y(1) = 0$. Considering the case studies $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 1.5$, we obtain respectively the solutions

$$\begin{aligned} y(z) &= 1 - 0.838447674076117z \\ &\quad - 0.0878743127704564z^2 - 0.0736780131534264z^3, \end{aligned} \tag{46}$$

$$y(z) = 1 - 0.7709169974z - 0.2290830032z^3, \tag{47}$$

$$\begin{aligned} y(z) &= 1 - 0.744115289348098z \\ &\quad + 0.207640336440601z^2 - 0.463525047092503z^3. \end{aligned} \tag{48}$$

It should be mentioned that the problem (20)-(21) has the exact solution (see discussion).

$$u = \frac{-\left(1-\beta T_2\right) + \sqrt{\left(1-\beta T_2\right)^2 + 2\beta H_1}}{\beta}, \tag{49}$$

where

$$H_1 = \frac{\beta T_1^2}{2} + \left(1-\beta T_2\right)T_1 + \frac{T_2 - T_1}{L}\left(1 - \frac{\beta}{2}\left(T_2 - T_1\right)\right)x. \tag{50}$$

6. Discussion

This work employed MHPMLT in the search for a handy accurate analytical approximate solution for the nonlinear second order ordinary differential equation with finite boundary conditions, which describes the steady state one-dimensional heat conduction in a slab with thermal conductivity, linearly dependent on the temperature. As it is well known, the temperature field of a body approaches asymptotically to steady state conditions; therefore the knowledge of the stationary solution is relevant because it determines the final temperature distribution along the slab. The case of thermal conductivity linearly dependent on the temperature arises, for instance, in the case of a pure metal such as copper. For this metal, at a temperature ranging from 0 to 5 K, the thermal conductivity is modelled mainly by electrons and increases linearly with temperature (see (22)) [27]. For other temperature ranges, this dependency is no longer linear, and should be the subject of future investigations, using for instance MHPMLT, to model the temperature distribution in these cases [27]. From (14) is inferred that MHPMLT is expressed in terms of the initial conditions for a given differential equation, therefore, our procedure was aimed to express the approximate solutions in terms of A [8], so that $y'(0)$ can be determined, just requiring that the approximate solution satisfies the boundary condition $y(1)=0$. This condition defines an algebraic equation for A, whose solution concludes the procedure by obtaining an analytical approximate solution for the proposed problem.

Figures 2-4 show the comparison between numerical solutions and approximate solutions (46)-(48) for $\varepsilon = 0.5$, $\varepsilon = 1$, and $\varepsilon = 1.5$ respectively. It can be noticed that curves are in good agreement, whereby it is inferred the potentiality of MHPMLT in the search for approximate solutions of nonlinear problems with finite boundary conditions [8]. Nevertheless, in more precise terms, it is possible to verify the accuracy of our results by calculating the square residual error (S.R.E) of approximate solutions (46)-(48). S.R.E is defined as $\int_a^b R^2\left(u(r)\right)dr$, where a and b are the end points, and $u(r)$ is an approximate solution to the equation to be solved (3), in our case (28) [9] and the residual $R(u(r))$ results of substituting $u(r)$ into differential equation to be solved. The resulting values were respectively of 0.0024, 0.0747 and 0.6745 which confirms the accuracy of the proposed solutions. The square residual error (S.R.E.) is in general terms a positive number, representative of the total error committed, by using the approximate solution $u(r)$ [9].

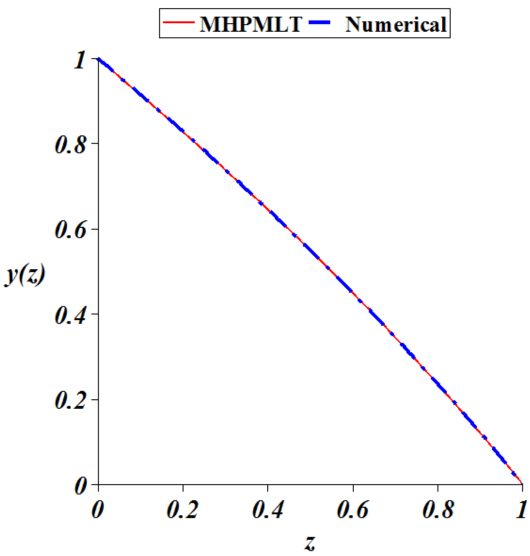

Figure 2. Comparison numerical solution of the nonlinear problem given by (28) and (29)) for and MHPMLT approximation (46).

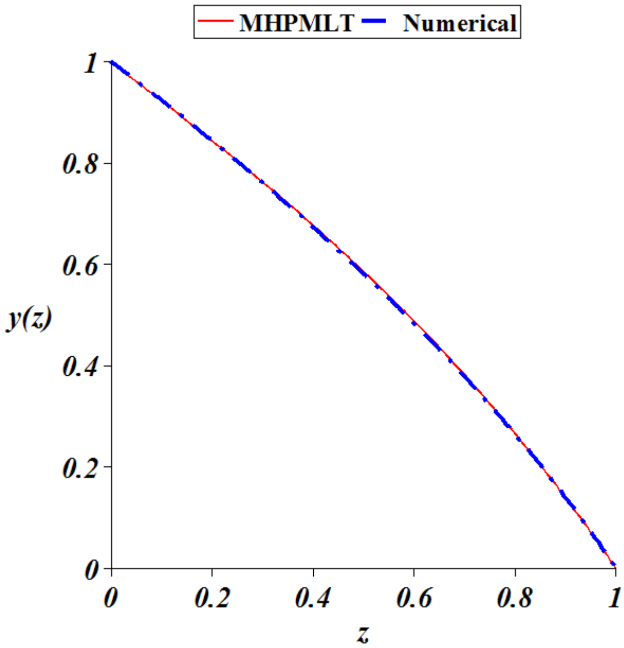

Figure 3. Comparison numerical solution of the nonlinear problem given by (28) and (29)) for and MHPMLT approximation (47).

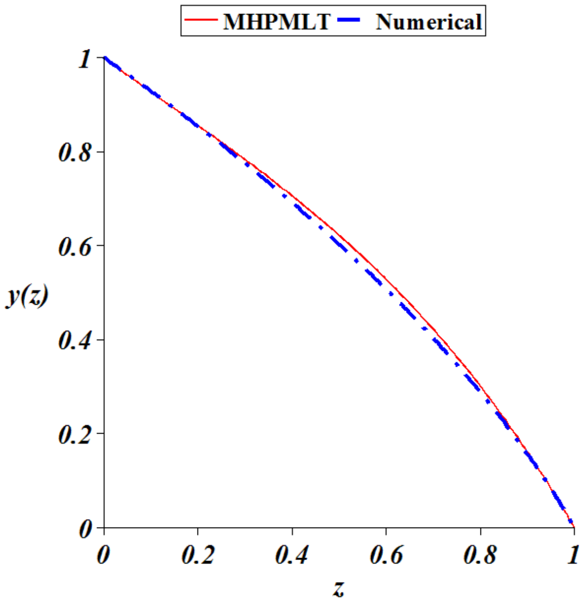

Figure 4. Comparison numerical solution of the nonlinear problem given by (28) and (29) for and MHPMLT approximation (48).

The parameter ε, turns out to be of paramount importance for our study. Thus, ε would be small for the case of small difference of temperature of the two opposite faces of the slab (see (24)) and for the same reason, for two given arbitrary points on the slab, this limit corresponds to the simpler case in which, thermal conductivity is almost constant (see (22)). From (28), it is deduced that for steady conditions, temperature varies linearly with z in this limit.

A more interesting case would occur if the temperature gradient along the slab was not necessarily small and corresponded to bigger values of ε. From (28), it is clear that the nonlinear term becomes important too. A relevant fact from MHPMLT follows from equations like (28), which can be written in the form $L(z)+\varepsilon N(z)=0$, where $L(z)$ is linear and $N(z)$ nonlinear [28]. It is well known that classical methods of approximation as perturbation method PM [28] provides in general, better results for small perturbation parameters $\varepsilon << 1$. From this point of view, ε can be visualized as a parameter of smallness that measures how greater is the contribution of linear term $L(z)$ than the one of $N(z)$. In general, it is easier to find analytical approximate solutions for small values of ε than for big values of the same. Figures 2-4 and the values of square residual error obtained show a noticeable fact, that (47) and (48) provide a good approximation as solutions of (28), despite of the fact that perturbation parameters $\varepsilon = 1$ and $\varepsilon = 1.5$ are indeed large. Thus, in principle, the proposed methodology is not restricted to small parameters [8] and it is able to explain the phenomenon under study for a wide range of values of the aforementioned parameter in a simple way.

On the other hand, it is very important to emphasize that, it is possible to improve the accuracy of our results (see Figure. 4), considering higher order approximate solutions.

In [9] optimal homotopy perturbation method (OHPM) was employed in order to provide an approximate solution for (28). Although the solution reported is handy and has good accuracy, this method is more complicated for applications than MHPMLT. OHPM requires construct a homotopy, which includes the presence of certain functions, provided of some parameters which are determined in order to control the convergence of solutions. This procedure is usually longer and difficult than MHPMLT which many times requires only of calculating elementary Laplace transforms in a systematic way.

At difference of other methods (for instance PM) which include the boundary conditions from the beginning of the problem at the lowest order approximation, MHPMLT estimates one of the initial conditions unknown at first, requiring that the whole proposed solution satisfies one of the boundary conditions (the other boundary condition is satisfied from the beginning of the procedure) thus, it is ensured that the approximate solution fits correctly on both boundaries of the interval (the above is provided, by calculating the value of A) [8].

Although this case study admitted the exact solution (49), it is necessary to make the following observations. We note that (49) and (50) provide a solution too long and somehow cumbersome for practical applications. Even though (46)-(48) are only approximate solutions for the propose problem, they are handy, accurate and for the same reason ideal for practical applications. Other theoretical and practical reasons related with the above, in favor of using the proposed method is that, the second order approximation (45) provided by MHPMLT is not only handy but it is expressed in terms of the perturbation parameter ε. Thus, from previously explained, at least for the case of moderated temperature gradients it is possible to employ (45) instead of the rather cumbersome and complicated exact solution (49) with small loss of precision. (45) is expressed in terms of physical parameter ε which allows in principle estimate the contributions of the different ε power terms. For example if ε is small so that we keep just the first two terms of (45), then the temperature varies very approximately in a linear way with z.

In consequence, this observation is even more important, regarding that the proposed method is not restricted to small parameters by which increases the fruitfulness of the proposed method because it is possible to employ (45) instead of (49) for a wider interval of ε. Finally, maybe the main reason for proposing MHPMLT, is that the majority of nonlinear differential equations that describe heat problems do not admit an exact solution.

7. Conclusions

In this work MHPMLT was employed to provide a handy analytical approximate solution for the second order nonlinear differential equation which describes the steady state one-dimensional heat conduction problem in a slab of thermal conductivity linearly dependent on the temperature defined with Dirichlet boundary conditions on a finite interval. MHPMLT method expressed the problem of finding an approximate solution for the above mentioned differential equation, in terms of solving an algebraic equation for some unknown initial condition [8]. The square residual error of the approximate solutions shows that it is a method with high potential in the search for solutions of boundary value nonlinear problems, even for the case of large perturbation parameters.

Finally, we note an important additional advantage from the proposed method; this does not require of solving several recurrence differential equations; indeed it requires only of calculating elementary Laplace transforms in a systematic way. Therefore, LTHPM is a tool with high potential for practical applications in science and engineering [8].

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