EXACT SOLUTION TO NONLINEAR BIOLOGICAL POPULATION MODEL WITH FRACTIONAL ORDER

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In this paper, Optimal Homotopy Asymptotic Method (OHAM) has been extended to seek out the exact solution of fractional generalized biological population models. The time fractional derivatives are described in the Caputo sense. It (OHAM) is a new approach for fractional models. The proposed approach presents a procedure by that we have transferred the model to a series of simpler problems which are solvable by hand work applying the Riemann-Liouville fractional integral operator and obtained exact solution of fractional the generalized biological population by adding the solutions of first three simple problems of the series of simpler problems. The new approach provides exact solution in the way of smoothly convergent series.

Keywords: Optimal Homotopy Asymptotic, Fractional generalized biological population, Exact solution.

1. Introduction

In recent years, the nonlinear fractional order partial differential equations (FPDEs) have been the center of attention of many studies due to their frequent applications in the fields of electromagnetic, electrochemistry, acoustics, material science, physics, viscoelasticity chemical processes, biology and engineering [1-5]. The solutions of nonlinear FPDEs have great interest both in mathematics and in useful applications. Therefore it is the main goal in the area of research of fractional models that how to develop a stable approach for to seek out the exact solution or approximate solution of FPDEs. Recently, several researchers have developed certain procedures for to seek out the analytical solution of nonlinear fractional order partial differential equations (FPDEs) like the method of variable separation, the Laplace transform Mellin transform, Fourier transform and other techniques [6-10]. These are those fractional differential equations who has the exact analytical solutions are only a few easy cases and equivalent to a few special functions such as the hyperbolic geometric function and the Fox H function [11,12] and the implementation of these methods for other strong nonlinear fractional models are very complicated. Some time the provided methods are not capable to control nonlinearity of FDEs to find out the solution FDEs. The reason of difficulties in finding exact solutions for most problems and the complexity of computing these special functions limit the applications of applied FDEs in engineering and scientific fields. In some cases the exact solutions of nonlinear FPDEs do not exist. Regarding these reasons, many of the scholars are searching and have developed numerical algorithms for approximate solution to solve the FDEs instead to find the exact solution to achieve the goal. These methods are the finite difference method, spectral element method and finite element method, [13-17] … etc. But it is the need of time to introduce a new method to achieve the goal which is ease in implementation and applicable for solution of all kind of nonlinear FDEs. Regarding this goal, we introduce a new recently developed method OHAM that how to construct the exact solution of nonlinear FDEs. In OHAM procedure we use homotopy's transformation in order to reduce nonlinear FDEs into simpler FDEs and construct solution by using logical stipulation of OHAM which rapidly converges to the exact solution. It has many successful applications into the integer order differential equations so far. OHPM has been used and introduced in many articles, see [19]. But in our work, the main focus is to extend a new approach OHAM for solving the exact solution of the nonlinear fractional-order biological population model [11]:

$$\frac{\partial^{\alpha} u(x,y,t)}{\partial t^{\alpha}} = \frac{\partial^2 u^2(x,y,t)}{\partial x^2} + \frac{\partial^2 u^2(x,y,t)}{\partial y^2} + f\left(u(x,y,t)\right), \qquad t > 0,\ \ 0 < \alpha \le 1 \tag{1}$$

subject to the initial condition:

$$u(x,y,0) = f_0(x,y),$$

where $u(x,y,t)$ denotes the population density and $f\left(u(x,y,t)\right) = u(x,y,t)\left(1 - r\,u(x,y,t)\right)$ represents the population supply due to births and deaths. The functions $u(X,t)$ and $f(u)$ describe the diffusion of a biological species in a region R and are the functions of position $X = (x,y)$ in R and time t. The function $u(X,t)$ shows the number of individuals, per unit volume, at position X and time t; its integral over any sub region shows the whole population of sub region at time t. The function $f(u)$ shows the average rate at that individuals are supplied (per unit volume) straightforwardly at X by births and deaths. For $\alpha \to 1$, the three examples of constitutive equations for $f(u)$ are:

$f(u) = c,$ where c is a constant, it leads to Malthusian Law,

$f(u) = u(r_1 - r_2 u),$ where r_1 and r_2 are positive constants, it leads to Verhulst Law,

$f(u) = -ru^p,\ (r \ge 0, 0 < p < 1)$, it leads to Porous Media, see [11].

The plan of the rest paper is as follows: Section 2 provides basic definition; Section 3 provides OHAM mathematical procedure; Implementation of OHAM to confirm convergence of OHAM solution to exact solution of selected fractional models is presented in Section 4. Finally, a brief conclusion and the further work have been listed.

2. Basic definitions:

We need the following axially results throughout the paper. Here we remark that the derivative has been taken in Caputo sense in this manuscript, see [18].

Definition 2.1. A real function $f(x)$, $x>0$, is said to be in space C_μ, $\mu \in \Re$, if there a real number $p > \mu$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$ and it is said to be in the space $C_\mu^{\ m}$ if only if $f^m \in C_\mu, m\in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f\in C_\mu, \mu \geq -1$ is defined as

$$J_{a_0}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{x} (x-\mu)^{\alpha-1} f(\mu) d\mu, \quad \alpha>0, \quad x>0, \tag{2}$$

$$J_{a_0}^{0} f(x) = f(x). \tag{3}$$

The Riemann-Liouville integral or fractional derivative operator have very useful and intresting applications in the real world problems. But, here, we are going to define caputo operators.

Definition 2.3. The fractional derivative of $f(x)$ in Caputo sense is defined as

$$D_{a_0}^{\alpha} f(x) = J_{a_0}^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a_0}^{x} (x-\eta)^{m-\alpha-1} f^m(\eta) d\eta, \tag{4}$$

where $m-1<\alpha\leq m, \ \ m\in N, \ x>0, \ f\in C_{-1}^{m}$

Definition 2.4. If $m-1<\alpha\leq m, \ \ m\in N$, and $f\in C_\mu^m, \ \mu\geq -1$, then

$$D_{a_0}^{\alpha} J_{a_0}^{\alpha} f(x) = f(x), \ \ D_{a_0}^{\alpha} J_{a_0}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)} \frac{(x-a_0)}{k!}, \quad x>0. \tag{5}$$

One can verify the operator J^α [1, 4] with the following useful and helpful properties:

For $f\in C_\mu^m, \ \alpha,\beta>0, \ \mu\geq -1 and \ \gamma \geq -1.$

- $J_{a_0}^{\alpha} f(x)$ exist for almost every $x\in[a_0,a_1]$.
- $J_{a_0}^{\alpha} J_{a_0}^{\beta} f(x) = J_{a_0}^{\alpha+\beta} f(x)$.
- $J_{a_0}^{\alpha} J_{a_0}^{\beta} f(x) = J_{a_0}^{\beta} J_{a_0}^{\alpha} f(x)$. (6)
- $J_{a_0}^{\alpha}(x-a_0)^{\gamma} = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a_0)^{\alpha+\gamma}$.

3. Mathematical formulation of OHAM for Fractional Order Models:

Consider the nonlinear fractional-order biological population Eq. (1) as:

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = A\big(u(x,y,t)\big) + f(x,y,t) \quad \alpha>0, \ (x,y)\in\Omega, \ t\in[a,b] \tag{7}$$

Subject to the initial condition

$$B\left(u(x,y,t), \frac{\partial u(x,y,t)}{\partial t}\right) = 0, \quad t\in\{a,b\}, \tag{8}$$

Here, $\dfrac{\partial^\alpha}{\partial t^\alpha}$ is described in the Caputo sense operator, $A\big(u(x,y,t)\big)$ is nonlinear terms in fractional order differential Eq.(7), $u(x,y,t)$ is unknown exact solution of Eq. (7) and $f(x,y,t)$ is given known function,

x, y and t are spatial and temporal independent variables respectively. Ω is domain, B is boundary operator of the Eq. (7).

Construct an optimal homotopy for unknown exact solution of the Eq. (7) as:

$\phi(x, y, t; p) : \Omega \times [0,1] \rightarrow R$

$$(1-p)\left(\frac{\partial^{\alpha}\phi(x,y,t)}{\partial t^{\alpha}} - f(x,y,t)\right) - H(t,p)\left(\frac{\partial^{\alpha}\phi(x,y,t)}{\partial t^{\alpha}} - \left(A(\phi(x,y,t)) + f(x,y,t)\right)\right) = 0, \tag{9}$$

here $p \in [0,1]$ is an embedding parameter, $(x, y) \in \Omega$ and $H(t, p)$ is non zero at $p \neq 0$ and $H(t,0) = 0$. when p increase in $[0,1]$, the homotopy ensures a rapid convergence of solution $\phi(x, y, t)$ to the exact solution of the Eq. (7). The accurate implementation of the OHAM, which does not depend on a small or large parameters appear in the nonlinear FPDEs, is purely based on the unbiased selection and true determination of the auxiliary function. $H(t, p)$ is an arbitrary chosen auxiliary function for Eq. (7). The region of fast convergence of the OHAM solution to the exact solution of Eq. (7) depends strictly on auxiliary function $H(t, p)$. Fundamentally, the expression of auxiliary function follows the terms appearing in nonlinear part $A(u(x, y, t))$ of the Eq. (7) such that product of the auxiliary function and $A(u(x, y, t))$ to be of the same shape.

The function $H(t, p)$ can be expressed as:

$$H(t,p) = pk_1(t,C_i) + p^2k_2(t,C_i) + p^3k_3(t,C_i) + \ldots \tag{10}$$

Where $C_i, i = 1,2,3,\ldots$ are auxiliary controller convergence constants and $k_i(t, C_i),\ i = 1,2,3,\ldots$ is a function of t and C_i. But to choose $k_i(t, C_i)$ is purely on the basis of terms appear in nonlinear part of the Eq. (7). Regarding this focal and logical point, we choose $k_1(t,C_i) = C_1,\ k_2(t,C_i) = C_2,\ k_3(t,C_i) = C_3 \ldots$ for nonlinear fractional-order biological population model because shape of the each solution of simpler problem will be same that is proved in simulation section.

By expanding $\phi(x, y, t; p, C_i)$ in Taylor's series about p, to seek out the exact solutions as:

$$\phi(x,y,t;C_i) = u_0(x,y,t) + \sum_{k=1}^{m} u_k(x,y,t;C_i)p^k, \qquad i = 1,2,3,\ldots \tag{11}$$

The point is to be noted that Eq. (11) converges to the exact solution of Eq. (7) at $p = 1$ as:

$$\tilde{u}(x,y,t;C_i) = u_0(x,y,t) + \sum_{k=1}^{\infty} u_k(x,y,t;C_i), \quad i = 1,2,3,\ldots \tag{12}$$

Generally speaking, we may truncate the Eq. (12) into finite terms to get the exact or approximate solution of nonlinear PDEs.

Equating co-efficient of the like powers of p by substituting Eq. (11) into Eq. (9), we get zero order, 1st order, 2nd order and high order problems.

$$\left.\begin{aligned}
&p^0: \frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} - f = 0,\\
&p^1: \frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} + (1+C_1)f + C_1 A_0 = 0,\\
&p^2: \frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha} - C_2\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha} + C_2(f+A_0) + C_1 A_1 = 0,\\
&p^3: \frac{\partial^\alpha u_3(x,y,t,C_1,C_2,C_3)}{\partial t^\alpha} - C_3\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} - C_2\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha} - (1+C_1)\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha} + C_3(f+A_0)\\
&\quad + C_2 A_1 + C_1 A_2 = 0\\
&\text{so on} \ldots
\end{aligned}\right\} \quad (13)$$

where $A_{k-j},\ j = 1,2,3\ldots k$ in the general kth-order governing problem is coefficient of the p^{k-j} with respect to the embed parameter p and

$$A\big(\phi(x,y,t;p,C_1,C_2,\ldots)\big) = A_0 + \sum_{i=1}^{\infty} A_i p^i.$$

Appling the J^α operator on the Eq. (13), we get series solutions:

$$u_0(r,t) = J^\alpha[f],$$

$$u_1(r,t;C_1) = J^\alpha\left[(1+C_1)\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} - (1+C_1)f - C_1 A_0\right],$$

$$u_2(r,t;C_1,C_2) = J^\alpha\left[C_2\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} + (1+C_1)\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha} - C_2(f+A_0) - C_1 A_1\right],$$

$$u_3(r,t;C_1,C_2,C_3) = J^\alpha\left[\begin{aligned}&C_3\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha} + C_2\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha} + (1+C_1)\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha}\\ &-C_3(f+A_0) - C_2 A_1 - C_1 A_2\end{aligned}\right],$$

so on . . . .

By substituting the above solutions in Eq.(12), one can get the exact solution $u(x,y,t;C_i)$ of the Eq.(7). The auxiliary convergence control constants $C_1, C_2, \ldots$ can be found by using least square method as:

$$\chi(C_i) = \int_0^t \iint_\Omega R^2(x,y,t;C_i)\,dxdydt,$$

$$\frac{\partial \chi}{\partial C_1} = \frac{\partial \chi}{\partial C_2} = \ldots\ldots = \frac{\partial \chi}{\partial C_n} = 0.$$

4. Numerical Simulation:

In this portion, we apply OHAM to get exact solution of the three special cases of the nonlinear fractional model Eq. (1) to show the accuracy and appropriateness of the new approach for to solving exactly nonlinear FPDEs.

Problem 1: Consider

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{\partial^2 u^2(x,y,t)}{\partial x^2} + \frac{\partial^2 u^2(x,y,t)}{\partial y^2} + u(x,y,t)\big(1 - r\,u(x,y,t)\big), \quad t>0,\ 0<\alpha\le 1, \qquad (14)$$

with the initial condition as:

$$u(x,y,0)=e^{\sqrt{\frac{r}{8}}(x+y)}. \tag{15}$$

The non linear part of Eq. (14)

$$A\left(u(x,y,t)\right)=\frac{\partial^2 u^2(x,y,t)}{\partial x^2}+\frac{\partial^2 u^2(x,y,t)}{\partial y^2}+u(x,y,t)\left(1-r\,u(x,y,t)\right).$$

Using the OHAM formulation we can get the following simpler problems with initial conditions as:

Zero-order problem:

$$\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}=0,\quad u_0(x,y,0)=e^{\sqrt{\frac{r}{8}}(x+y)}. \tag{16}$$

First-order problem:

$$\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha}-(1+C_1)\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}+C_1A_0=0,\ u_1(x,y,t)=0. \tag{17}$$

Second-order problem:

$$\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha}-C_2\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}-(1+C_1)\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha}+C_2A_0+C_1A_1=0,\quad u_2(x,y,t)=0. \tag{18}$$

Third-order problem:

$$\left.\begin{aligned}&\frac{\partial^\alpha u_3(x,y,t,C_1,C_2,C_3)}{\partial t^\alpha}-C_3\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}-C_2\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha}-(1+C_1)\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha}+C_3A_0+C_2A_1+C_1A_2=0,\\&u_3(x,y,t)=0.\end{aligned}\right\} \tag{19}$$

Respective solutions of the Eq.(16)-Eq.(19) are given as follow:

$$u_0(x,y,t)=e^{\sqrt{\frac{r}{8}}(x+y)},$$

$$u_1(x,y,t,C_1)=-C_1e^{\sqrt{\frac{r}{8}}(x+y)},$$

$$u_2(x,y,t,C_1,C_2)=-B_0e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^\alpha}{\Gamma(\alpha+1)}+C_1^{\,2}e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,y,t,C_1,C_2,C_3)=-B_1e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^\alpha}{\Gamma(\alpha+1)}+2B_0C_1e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-C_1^{\,3}e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

So on….

In similar way, we can compute the solution of next simpler problems.

Finally, we can get the following expression:

$$\tilde{u}(x,y,t)=u_0(x,y,t)+u_1(x,y,t,C_1)+u_2(x,y,t,C_1,C_2)+u_3(x,y,t,C_1,C_2,C_3)\ \ldots$$

$$\tilde{u}(x,y,t)=e^{\sqrt{\frac{r}{8}}(x+y)}-(C_1+B_0+B_1)e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^\alpha}{\Gamma(\alpha+1)}+(C_1^{\,2}+2C_1B_0)e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-C_1^{\,3}e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\ \ldots \tag{20}$$

where

$$B_0=C_2+C_1(1+C_1),$$

$$B_1=C_3+C_2C_1+C_2(1+C_1)+C_1(1+C_1)^2.$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1=-1.0000356780000000000,$$

$$C_2=-0.0000000004663021896,$$

$$C_3=6.583711972000000\times10^{-14}.$$

by substituting the values of auxiliary constants into the Eq. (20), we obtain the expression

$$\tilde{u}(x, y, t) = e^{\sqrt{\frac{r}{8}}(x+y)} + e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + e^{\sqrt{\frac{r}{8}}(x+y)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \cdots$$

$$\tilde{u}(x, y, t) = e^{\sqrt{\frac{r}{8}}(x+y)}\sum_{i=1}^{\infty}\frac{t^{i\alpha}}{\Gamma(i\alpha+1)} = e^{\sqrt{\frac{r}{8}}(x+y)}E_{\alpha}\left(t^{\alpha}\right).$$

If $\alpha = 1$ then we get exact solution of the Eq. (14)-Eq. (15):

$$\tilde{u}(x, y, t) = e^{\sqrt{\frac{r}{8}}(x+y)+t}.$$

It is full agreed with the exact solution .

Problem 2: Consider

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u^2(x,y,t)}{\partial x^2} + \frac{\partial^2 u^2(x,y,t)}{\partial y^2} + hu(x,y,t), \qquad t > 0,\ 0 < \alpha \le 1 \tag{21}$$

Subject to the initial condition:

$$u(x,0) = \sqrt{xy}. \tag{22}$$

The non linear part of Eq. (21):

$$A\left(u(x,y,t)\right) = \frac{\partial^2 u^2(x,y,t)}{\partial x^2} + \frac{\partial^2 u^2(x,y,t)}{\partial y^2} + hu(x,y,t).$$

Using the OHAM formulation we can get the following simpler problems with initial conditions as:

Zero order problem:

$$\frac{\partial^{\alpha}u_0(x,y,t)}{\partial t^{\alpha}} = 0, \qquad u_0(x,y,0) = \sqrt{xy}. \tag{23}$$

First-order problem:

$$\frac{\partial^{\alpha}u_1(x,y,t,C_1)}{\partial t^{\alpha}} - (1+C_1)\frac{\partial^{\alpha}u_0(x,y,t)}{\partial t^{\alpha}} + C_1 A_0 = 0,\ u_1(x,y,t) = 0. \tag{24}$$

Scond-order problem:

$$\frac{\partial^{\alpha}u_2(x,y,t,C_1,C_2)}{\partial t^{\alpha}} - C_2\frac{\partial^{\alpha}u_0(x,y,t)}{\partial t^{\alpha}} - (1+C_1)\frac{\partial^{\alpha}u_1(x,y,t,C_1)}{\partial t^{\alpha}} + C_2 A_0 + C_1 A_1 = 0, \quad u_2(x,y,t) = 0. \tag{25}$$

Third-order problem:

$$\left.\begin{aligned}&\frac{\partial^{\alpha}u_3(x,y,t,C_1,C_2,C_3)}{\partial t^{\alpha}} - C_3\frac{\partial^{\alpha}u_0(x,y,t)}{\partial t^{\alpha}} - C_2\frac{\partial^{\alpha}u_1(x,y,t,C_1)}{\partial t^{\alpha}} - (1+C_1)\frac{\partial^{\alpha}u_2(x,y,t,C_1,C_2)}{\partial t^{\alpha}} + C_3A_0 + C_2A_1 + C_1A_2 = 0,\\ &u_3(x,y,t) = 0.\end{aligned}\right\} \tag{26}$$

The respective solutions of the Eq.(23)-Eq.(26) are given as follow:

$$u_0(x,y,0) = \sqrt{xy}.$$

$$u_1(x,y,t,C_1) = -C_1\sqrt{xy}.$$

$$u_2(x,y,t,C_1,C_2) = -B_0\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + C_1^{\ 2}\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$u_3(x,y,t,C_1,C_2,C_3) = -B_1\sqrt{xy}\frac{t^{\alpha}}{\Gamma(\alpha+1)} + 2C_1B_0\sqrt{xy}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - C_1^{\ 3}\sqrt{xy}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

So on….

In similar way, we can compute the solution of next simpler problems.

Finally we can get the following expression:

$$\tilde{u}(x, y, t) = u_0(x, y, t) + u_1(x, y, t, C_1) + u_2(x, y, t, C_1, C_2) + u_3(x, y, t, C_1, C_2, C_3).....$$

$$\tilde{u}(x, y, t) = \sqrt{xy} - (C_1 + B_0 + B_1)\sqrt{xy}\,\frac{(ht)^{\alpha}}{\Gamma(\alpha+1)} + (C_1^2 + 2C_1B_0)\sqrt{xy}\,\frac{(ht)^{2\alpha}}{\Gamma(2\alpha+1)} - C_1^3\sqrt{xy}\,\frac{(ht)^{3\alpha}}{\Gamma(3\alpha+1)}... \tag{27}$$

where

$$B_0 = C_2 + C_1(1 + C_1),$$
$$B_1 = C_3 + C_2C_1 + C_2(1 + C_1) + C_1(1 + C_1)^2.$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1 = -1.0003095680000000,$$
$$C_2 = 0.00000002526788734,$$
$$C_3 = 2.35014404700 \times 10^{-11}.$$

by substituting the values of auxiliary constants into the Eq. (27), we obtain the expression

$$\tilde{u}(x, y, t) = \sqrt{xy} + \sqrt{xy}\,\frac{(ht)^{\alpha}}{\Gamma(\alpha+1)} + \sqrt{xy}\,\frac{(ht)^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{(ht)^{3\alpha}}{\Gamma(3\alpha+1)} ....$$

$$\tilde{u}(x, y, t) = \sqrt{xy}\sum_{i=1}^{\infty}\frac{(ht^{\alpha})^{i}}{\Gamma(i\alpha+1)} = \sqrt{xy}\,E_{\alpha}(ht^{\alpha}).$$

If $\alpha = 1$ then we get exact solution of the Eq. (21)-Eq. (22):

$$\tilde{u}(x, y, t) = \sqrt{xy}\,e^{ht}.$$

It is full agreed with the exact solution.

Problem 3: Consider the nonlinear time-fractional Advection equation

$$\frac{\partial^{\alpha} u(x, y, t)}{\partial t^{\alpha}} = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t), \quad t > 0,\ 0 < \alpha \le 1, \tag{28}$$

with initial condition as:

$$u(x, y, 0) = \sqrt{\sin(x)\sinh(y)}. \tag{29}$$

The non linear part of Eq. (28):

$$A(u(x, y, t)) = \frac{\partial^2 u^2(x, y, t)}{\partial x^2} + \frac{\partial^2 u^2(x, y, t)}{\partial y^2} + u(x, y, t).$$

Using the OHAM formulation we can get the following simpler problems with initial conditions as:

Zero-order problem:

$$\frac{\partial^{\alpha} u_0(x, y, t)}{\partial t^{\alpha}} = 0, \qquad u_0(x, y, 0) = \sqrt{\sin(x)\sinh(y)}. \tag{30}$$

First-order problem:

$$\frac{\partial^{\alpha} u_1(x, y, t, C_1)}{\partial t^{\alpha}} - (1 + C_1)\frac{\partial^{\alpha} u_0(x, y, t)}{\partial t^{\alpha}} + C_1A_0 = 0,\ u_1(x, y, t) = 0. \tag{31}$$

Second-order problem:

$$\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha}-C_2\,\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}-(1+C_1)\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha}+C_2A_0+C_1A_1=0,\quad u_2(x,y,t)=0. \tag{32}$$

Third-order problem:

$$\left.\begin{aligned}&\frac{\partial^\alpha u_3(x,y,t,C_1,C_2,C_3)}{\partial t^\alpha}-C_3\frac{\partial^\alpha u_0(x,y,t)}{\partial t^\alpha}-C_2\,\frac{\partial^\alpha u_1(x,y,t,C_1)}{\partial t^\alpha}-(1+C_1)\frac{\partial^\alpha u_2(x,y,t,C_1,C_2)}{\partial t^\alpha}\\&+C_3A_0+C_2A_1+C_1\,A_2=0,\ u_3(x,y,t)=0.\end{aligned}\right\} \tag{33}$$

The respective solutions of Eq.(30)-Eq.(33) are given as follow:

$$u_0(x,y,0)=\sqrt{\sin x\sinh y}$$

$$u_1(x,y,t)=-C_1\sqrt{\sin x\sinh y}$$

$$u_2(x,y,t)=-B_0\sqrt{\sin x\sinh y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}+C_1^{\,2}\sqrt{\sin x\sinh y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,y,t)=-B_1\sqrt{\sin x\sinh y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}+2C_1B_0\sqrt{\sin x\sinh y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}-C_1^{\,3}\sqrt{\sin x\sinh y}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}.$$

so on….

In similar way, we can compute the solution of next simpler problems.

Finally, we can get the following the expression:

$$\tilde{u}(x,y,t)=u_0(x,y,t)+u_1(x,y,t,C_1)+u_2(x,y,t,C_1,C_2)+u_3(x,y,t,C_1,C_2,C_3).....$$

$$\left.\begin{aligned}\tilde{u}(x,y,t)=&\sqrt{\sin x\sinh y}-(C_1+B_0+B_1)\sqrt{\sin x\sinh y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}+(C_1^{\,2}+2C_1B_0)\sqrt{\sin x\sinh y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\&-C_1^{\,3}\sqrt{\sin x\sinh y}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\cdots\end{aligned}\right\} \tag{34}$$

where

$$B_0=C_2+C_1(1+C_1),$$
$$B_1=C_3+C_2C_1+C_2(1+C_1)+C_1(1+C_1)^2.$$

After finding the residual, we used least square method and then obtained the values of auxiliary constants:

$$C_1=-0.9999127255000000000,$$
$$C_2=-0.0000000002378853737,$$
$$C_3=-1.241770350000\times10^{-13}.$$

By substituting the values of auxiliary constants into the Eq. (34), we obtain

$$\tilde{u}(x,y,t)=\sqrt{\sin x\sinh y}+\sqrt{\sin x\sinh y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}+\sqrt{\sin x\sinh y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\sqrt{\sin x\sinh y}\,\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\ ...$$

$$\tilde{u}(x,y,t)=\sqrt{\sin x\sinh y}\sum_{i=1}^{\infty}\frac{(t^\alpha)^i}{\Gamma(i\alpha+1)}=\sqrt{\sin x\sinh y}\,E_\alpha(t^\alpha).$$

If $\alpha=1$ then we get the exact solution of the Eq. (21)-Eq. (22):

$$\tilde{u}(x,y,t)=\sqrt{\sin x\sinh y}\,e^t,$$

Which is full agreed with the solution.

Conclusion

Exact solution of nonlinear fractional order biological population model with time fractional derivatives has been obtained by successful application of new approach OHAM. In the fields of applied mathematics generally in the recent emergence of nonlinear fractional differential equations, it is necessary to study various methods to obtain the solutions of such nonlinear FPDEs. That are why we expect that our work is a pace towards to obtain the exact solution of physical problems devoted to nonlinear FPDEs. In simulation section of our work, it has been proved that the exact solutions of nonlinear fractional order biological population model obtained by extended new approach are in admirable agreement with the exact solutions of nonlinear fractional order biological population model [49]. The application of this new approach into solving every kind of nonlinear time-fractional and space-fractional partial or ordinary differential equations will be our further consideration.

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