ON DISSIPATIVE MHD MIXED CONVECTION BOUNDARY-LAYER FLOW OF JEFFREY FLUID OVER AN INCLINED STRETCHING SHEET WITH NANOPARTICLES: Buongiorno Model

by

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Present paper utilizes a combination of non-Newtonian fluid model (Jeffrey fluid) with Buongiorno model (nanofluid). The Jeffery fluid, which is regarded as a base fluid, together with suspended nanoparticles are examined over an inclined stretching sheet with the amalgamated impacts of mixed convection and viscous dissipation. The mathematical formulation of this model is done by choosing the appropriate similarity variables for the aim to reduce the complexity of governing partial differential equations. The Runge-Kutta-Fehlberg (RKF45) method is then applied to the resulting of non-linear ODE to generate numerical results for highlighting the impact of emerging parameters towards specified distributions. Both the graphical and tabular representations of vital engineering physical quantities are also shown and deliberated. For the increase of Eckert number, thermophoresis diffusion, and Brownian motion parameters, the elevation of temperature profiles is observed. Besides, the thermophoresis diffusion parameter tends to accelerate the nanoparticle concentration profile while Brownian motion parameter displays the opposite behavior.

Key words: Jeffrey nanofluid, prescribed wall temperature, viscous dissipation, inclined stretching sheet, Buongiorno model

Introduction

The enlargement in industrial and technological applications has increased the need for knowledge on the concept of heat and mass transfer, particularly on the non-linear rheological fluid. In fact, this concept is vital as it contributes to the success of numerous engineering applications. These engineering applications may consist of the manufacturing of plastic and rubber sheets, production of glass fibers, extrusion process, polymer technology etc. Recent investigations have shown that the non-linear rheological fluid had received extra attention [1-5]. This is due to the complex nature of fluid used in most industrial applications, that a single constitutive equation is inadequate to describe such fluids. Differs to Newtonian fluid, the connection between the rate of stress and strain is non-linear because of dependency of

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fluid viscosity on time or deformation. According to the literature, many non-Newtonian fluids have been proposed; yet, each of them displays dissimilar characteristics. Therefore, examining these fluids entirely at one time is slightly impossible. In view of the special ability in explaining the physical characteristics of both relaxation and retardation times, the Jeffrey fluid model has been picked for further research. Thorough investigations related to this fluid model include those of Dalir [6], Hayat et al. [7], Narayana and Babu [8], Ojjela et al. [9], Hayat et al. [10], and Hayat et al. [11].

It is known that nanofluids are the solid-liquid composite materials containing suspended nanoparticles such as oxides, metals, nitrides, carbides, or nonmetals, i.e. graphite and carbon nanotubes with diameter between 1 and 100 nm. The properties of the base fluid, such as water or ethylene glycol are conceivably enhanced by mixing it with nanoparticles; nonetheless, subjected to the particle’s shape, size and the base fluid itself [12]. Here, the presence of nanoparticles is significant as it can strongly enhance the thermal properties as well as is highly potential to the rate of heat exchange with no drop in pressure. Moreover, the Brownian motion and thermophoresis diffusion are liable on the enhancement of the thermal performance [13]. Accordingly, nanofluids have undeniably become a subject of interest to be studied. A comparison study on the combination of ethylene glycol with copper nanoparticles and pure ethylene glycol or ethylene glycol with oxide nanoparticles was explored by Eastman et al. [14] to examine the thermal properties of each fluid. In comparison to the ethylene glycol with oxide nanoparticles, it is evident that the thermal properties of ethylene glycol and copper nanoparticles are greater. Choi et al. [15] discovered a distinguished change in the thermal performance of the base fluid as a result of dispersion of a very small quantity of nanotubes. Precisely, with 1 vol.% of nanotubes, the ratio of thermal conductivity is greater than 2.5. Khan and Pop [16] were the first who studied the numerical solution of nanofluid passing through a stretching sheet. With the aim of focusing on the same problem, Makinde and Aziz [17] extended the study to the convective boundary conditions. The stagnation point flow of power-law nanofluid from a convectively heated stretching sheet was then tackled by Ibrahim and Makinde [18]. Following the Buongiorno model, the Jeffrey fluid model with the nanoparticles were employed by Shehzad et al. [19], Avinash et al. [20], and Qayyum et al. [21] to investigate the influences of Jeffrey fluid, Brownian motion and thermophoresis diffusion parameters.

The involvement of the traits from the forced and free flow conditions, or mixed convection flow is pronounced when the forced flow impact in the free convection or the buoyancy force impact in the forced convection, is sizable. In particular, the buoyancy parameter, $\gamma = \frac{Gr}{Re^2_{x}}$ confers a measure of free convection to the forced convection flows effects, where $\gamma \rightarrow 0$ signifies the dominant mode of transport for the forced convection, whereas $\gamma \rightarrow \infty$ implies the dominant mode of free convection. However, the contribution of each convections on heat transfer enhancement is reliant on the fluid-flow, temperature, orientation and surface geometry. Abbasi et al. [22] scrutinized the mixed convection flow with thermal radiation and double stratification effects in MHD Jeffrey nanofluid induced by stretching sheet. They noted that the changes in the temperature profile for the smaller values of mixed convection parameter was still significant compared to the concentration buoyancy parameter. In other work, Dhanai et al. [23] inspected the impacts of velocity and thermal slip past an inclined cylinder with mixed convection. Aided by the Runge-Kutta-Fehlberg method together with shooting technique, the free convection flow was revealed to be more effective than the forced convection flow for larger value of mixed convection parameter, and this implies a rise to the fluid-flow.
The MHD reflects the science of the motion of electrically conducting fluids under magnetic fields. Conceptually, magnetic fields can instigate currents in a moving conductive fluid, that result in the polarization of fluid and concurrently modify the magnetic field. This situation is principally one of the reciprocal contacts between the motion of fluid and the magnetic field; the fluid-flow alters the magnetic field and vice versa. Problem concerning the heat transfer and electrically conducting flow past a stretching sheet has turned into one of the most favorable discussed topics. The paramount interest of the topic can be affiliated to the continuously growing industrial applications, specifically in the productions of cable coatings, glassware, paper, fabrications of adhesive tapes, metal spinning, metallic plates and fine-fiber mats. The stretching and cooling processes of such industrial applications need to be controlled to achieve desirable manufactured products. In essence, the manufacturing process requires both the thermal and mechanical interactions of stretched surface with the ambient fluid. A pioneering study was conducted by Sakiadis [24] on the boundary-layer flow over a continuously moving surface. The flow past a stretching plate was subsequently enquired into by Crane [25]. Henceforth, various researchers elongated this research area by addressing different surface geometry with several effects [26-29]. However, the current study will be restricted to the inclined stretching sheet. Reddy [30] discussed the combined impacts of thermal radiation and chemical reaction in MHD Casson fluid past an exponentially inclined permeable stretching sheet. Afridi et al. [31] examined the impact of entropy generation in a viscous fluid from an inclined stretching sheet. Approximate analytical approach, namely homotopy analysis method was executed by Sravanthi [32] to observe the influences of suction/blowing, Soret and Dufour and thermal radiation over an exponentially inclined stretching sheet. Other distinguished explorations allied with the inclined stretching sheet were published by Thumma et al. [33] and Gupta et al. [34].

A noteworthy effect of viscous dissipation is discerned in the case of highly viscous fluid or when the fluid-flows very fast. Heat is basically generated from the frictional force induced by the shear in the fluid-flow. Soundalgekar [35] studied the viscous dissipation effect with constant suction due to a vertical porous plate. The same effect was then considered by Yirga and Shankar [36], Mohamed et al. [37, 38], and Besthapu et al. [39] in a nanofluid past a stretching sheet, horizontal circular cylinder and exponentially stretching sheet, respectively. Meanwhile, Hussain et al. [40] addressed this effect from a stretching cylinder embedded in the MHD Sisko nanofluid. Recently, the combined effects of viscous dissipation and Joule heating were scrutinized by Kumar et al. [41] and Mahanthesh and Gireesha [42] in the respective ferro-nano-fluid and two phase flow of Casson fluid.

Based on the aforementioned investigations, it is ascertainable that no endeavours has been effectuated on the impact of viscous dissipation in MHD mixed convection Jeffrey fluid with nanoparticles due to the inclined stretching sheet. The model of Jeffrey fluid is herein considered as a base fluid. With the succor of MAPLE package, the RKF45 method is embarked upon to solve the highly non-linear ODE. The impacts of emerging parameters towards specified distributions are highlighted in the graphical and tabular forms. As yet, there has been no prevailing study published on this topic, thus the authors acknowledge this work as new.

**Problem formulation**

A steady, 2-D and laminar flow of mixed convection boundary-layer flow in an electrically conducting Jeffrey nanofluid past an inclined stretching sheet with an inclination angle, \( \alpha_0 \), to the vertical is deliberated. Here, the velocity of the stretching sheet is considered linear, i.e., \( u_w(x) = ax \) (\( a \) is constant and \( a > 0 \)). The stretched surface of temperature and
nanoparticle concentrations is further considered to be greater than the free stream of temperature and nanoparticle concentration where \( T_u(x) > T_\infty \) and \( C_u(x) > C_\infty \), respectively. The transverse magnetic field, \( B_0 \), is imposed perpendicularly to the stretched surface. Figure 1 displays the schematic diagram of the problem, in which \( x \)- and \( y \)-axes are directed in the same direction of the flow field and normal to it, respectively. To obtain the self-similarity equation, the thermal and nanoparticle concentration expansion coefficients are assumed to be \( \beta_T = mx^{-1} \) and \( \beta_C = nx^{-1} \), whereby \( m \) and \( n \) are constants \([43, 44]\) while the temperature and nanoparticle concentration of the stretching sheet are \( T_u(x) = T_\infty + bx^2 \) and \( C_u(x) = C_\infty + dx^2 \), where \( b \) and \( d \) are constants \([31, 45]\). The governing equations of Jeffrey nanofluid can be expressed \([6, 22]\):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\frac{u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\nu}{1+\lambda} \left[ \frac{\partial^2 u}{\partial y^2} + \lambda \left( \frac{\partial u}{\partial y} \right)^2 \right] - \frac{\sigma B_0^2}{\rho_f} u +
\]

\[
+ \left[ \beta_T \rho_f (T - T_\infty) + \beta_C (C - C_\infty) \right] g \cos \alpha_0
\]

\[
\frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\alpha}{\nu} \frac{\partial^2 T}{\partial y^2} + \tau \left[ D_B \frac{\partial C}{\partial y} + D_f \left( \frac{\partial T}{\partial y} \right)^2 \right] +
\]

\[
+ \frac{\nu}{C_p (1+\lambda)} \left[ \left( \frac{\partial u}{\partial y} \right)^2 \right] + \lambda \left( \frac{\partial u}{\partial y} \right)^2 + v \frac{\partial^2 u}{\partial y^2}
\]

\[
\frac{\alpha}{\nu} \frac{\partial C}{\partial x} + \frac{\partial C}{\partial y} = D_B \frac{\partial^2 C}{\partial y^2} + D_f \frac{\partial^2 T}{\partial y^2}
\]

The associated boundary conditions are:

\[
u = u(x) = ax, \quad v = 0, \quad T = T_u(x) = T_\infty + bx^2, \quad C = C_u(x) = C_\infty + dx^2 \quad \text{at} \quad y = 0
\]

\[
u \to 0, \quad T \to T_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty
\]

From the equations, the respective velocity components in the \( x \)- and \( y \)-directions are written as \( u \) and \( v \) while \( \alpha, \nu, \rho_f, \beta_T, \beta_C, T, C, D_B, D_f \) being the respective thermal diffusivity, gravitational acceleration, kinematic viscosity, electrical conductivity, density of fluid, ratio of relaxation to retardation times, retardation time, fluid temperature, nanoparticle concentration of the fluid, specific heat capacity, Brownian motion, and thermophoretic diffusion parameters. In addition, let \( \tau = (\rho c_p)_f/(\rho c_p)_f \) be the ratio of nanoparticle heat capacity to the base fluid heat capacity. The following similarity transformation variables are imposed in order to reduce the complexity of eqs. (1)-(4) \([16, 22] \).
\begin{align*}
\eta &= (a/v)^{1/2} y, \quad u = axf'(\eta), \quad v = -(av)^{1/2} f(\eta), \\
\theta(\eta) &= (T - T_\infty)/(T_w - T_\infty), \quad \phi(\eta) = (C - C_\infty)/(C_w - C_\infty) \quad (6)
\end{align*}

On account of the transformations, eq. (1) is inevitably satisfied and eqs. (2)-(4) become:

\begin{align*}
f'' + \lambda_2 (f'' - ff') - (1 + \lambda)(f'' - f) - (1 + \lambda) Mf' + (1 + \lambda) \gamma (\theta + N\phi) \cos \alpha_0 &= 0 \quad (7) \\
(1 + \lambda) \theta'' + (1 + \lambda) \Pr (f \theta' - 2 f \theta + Nb \theta' \phi' + N \theta'^2) + \\
&+ \Ec Pr[f'^2 + \lambda_2 (f'^2 - ff')] = 0 \\
\phi'' + \Le Pr[f \phi'' - 2 f \phi] + (N/t Nb) \theta'' &= 0 \quad (8)
\end{align*}

subject to the boundary conditions:

\begin{align*}
f(0) &= 0, \quad f'(0) = 1, \quad \theta(0) = 1, \quad \phi(0) &= 1 \quad (10)
\end{align*}

The dimensionless parameters arising in eqs. (7)-(9) are:
- \( \Pr / \nu \alpha = - \text{the Prandtl number} \)
- \( \gamma = GR/Re^2 = g \beta_f (T_w - T_\infty) x u_w^2 \) - the mixed convection parameter,
- \( M = \sigma B_0^2 / \rho \sigma \) - the magnetic parameter,
- \( \lambda_2 = \lambda_2 a = - \text{the Deborah number} \)
- \( \Le = \alpha / D_B = - \text{the Lewis number} \)
- \( N = \beta_C (C_w - C_\infty) / \beta_T (T_w - T_\infty) = - \text{the concentration buoyancy parameter} \)
- \( Nb = (\rho C)_p D_B / (C_w - C_\infty) = - \text{the Brownian motion parameter} \)
- \( \Ec = u_w^2 / (\rho C)_p (T_w - T_\infty) = - \text{the Eckert number} \)
- \( \Nt = (\rho C)_p D_T (T_w - T_\infty) / (\rho C)_p v T_\infty = - \text{the thermophoresis diffusion parameter} \)

The exact solution of eq. (7) when \( M = \gamma = 0 \) is given as [7]:

\begin{align*}
f(\eta) &= (1 - e^{-\eta})/r \\
\phi''(0) &= -r \\
\theta'(x) &= 0, \quad \phi'(x) &= 0\quad (11)
\end{align*}

where \( r = [(1 + \lambda)/(1 + \lambda_2)]^{1/2} \). The reupon, \( f''(\eta) = -re^{-\eta} \) denotes the second derivative of eq. (11). At the surface of the sheet, the gradient of velocity is:

\begin{align*}
f''(0) &= -r \quad (12)
\end{align*}

The requisite engineering physical quantities such as the local skin friction coefficient, Nusselt number and Sherwood number are:

\begin{align*}
C_f &= \frac{\tau_w}{1/2 \rho u_w^2(x)}, \quad \text{Nu}_x = \frac{xq_w}{k(T_w - T_\infty)} \quad \text{and} \quad \text{Sh}_x = \frac{xq_m}{D_B (C_w - C_\infty)} \quad (13)
\end{align*}

where

\begin{align*}
\tau_w &= \frac{\mu}{1 + \lambda} \left[ \frac{\partial u}{\partial y} + \lambda_1 \left( u \frac{\partial^2 u}{\partial x \partial y} + v \frac{\partial^2 u}{\partial y^2} \right) \right]_{y=0}, \quad q_w = -k \frac{\partial T}{\partial y}_{y=0}, \quad \text{and} \quad q_m = -D_B \frac{\partial C}{\partial y}_{y=0}
\end{align*}

are the shear stress, surface heat flux, and surface mass flux, respectively. The reduced engineering physical quantities which consist of skin friction coefficient, Nusselt number and Sherwood number are:
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\[ C_f \, \text{Re}_x^{1/2} = \frac{2}{1 + \lambda} \left[ f''(0) + \lambda_2 f''(0) \right], \quad \text{Nu}_x \, \text{Re}_x^{-1/2} = -\theta'(0), \quad \text{and} \quad \text{Sh}_x \, \text{Re}_x^{-1/2} = -\phi'(0) \]  

(14)

where the local Reynolds number is defined as \( \text{Re}_x = \frac{u_x x}{v} \).

**Particular case:** In the nonappearance of mixed convection parameter \( \gamma = 0 \) and magnetic parameter \( M = 0 \), the present solution of momentum eq. (7) excellently coincides with that of Dalir [6], whose results are congruous with the exact solution of eq. (12). Besides, eqs. (7) and (8) are reduced to the Newtonian model proposed by Afridi et al. [31], provided that the absenteeism of the following parameters: \( \lambda = \lambda_2 = N = Nb = Nt = Le = 0 \).

**Numerical procedure**

Equations (7)-(9) together with boundary conditions (10) are solved numerically using the RKF45 method with the aid of MAPLE package. This method implicates the transformation of eqs. (7)-(9) into first order system by supposing \( f, f', f'', f''', \theta, \theta', \phi, \phi' \) = \( (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8) = Z \), as presented below [23, 46]:

\[
\begin{pmatrix}
Z'_1 \\
Z'_2 \\
Z'_3 \\
Z'_4 \\
Z'_5 \\
Z'_6 \\
Z'_7 \\
Z'_8
\end{pmatrix}
= \begin{pmatrix}
Z_2 \\
Z_3 \\
Z_4 \\
Z_5 \\
Z_6 \\
Z_7 \\
Z_8 \\
\end{pmatrix} - \frac{1}{Z_1} \left[ Z_2^2 - \frac{1}{\lambda_2} \left[ (1 + \lambda)(Z_2^2 - Z_2 Z_3) + (1 + \lambda)MZ_2 - (1 + \lambda)\gamma(Z_4 + NZ_7)\cos \alpha_0 - Z_4 \right] \right] \\
- \text{Pr} \left( Z_1 Z_6 - 2Z_2 Z_5 + Nb Z_6 Z_8 + Nt Z_8^2 \right) - \frac{\text{Ec Pr}}{(1 + \lambda) \left[ Z_3^2 + \lambda_2 Z_3 \left( Z_2 Z_3 - Z_4 \right) \right]} \\
\frac{Nt}{Nb} Z_6' - \text{Le Pr} \left( Z_1 Z_8 - 2Z_2 Z_7 \right)
\end{pmatrix}
\]

(15)

with the initial conditions:

\[
Z^T = (0, 1, Z_3, 1, Z_6, 1, Z_8)^T
\]

The system utilizes the Newton-Raphson method and shooting method to guess the missing initial conditions, \( Z_1, Z_4, Z_6, \) and \( Z_8 \) i.e. \( f''(0), f'''(0), \theta'(0), \) and \( \phi'(0) \) by an iterative process until the boundary conditions are satisfied. The convergence criteria is fixed to be \( 10^{-6} \), while the boundary-layer thickness is set from \( \eta_0 = 5 \) to 10 to acquire the asymptotic behaviours of velocity, temperature and concentration profiles.

**Results and discussion**

A comprehensive solution of the dimensionless parameters on the velocity \( f''(0) \), temperature, \( \theta(0) \), and nanoparticle concentration \( \phi(0) \) profiles as well as the reduced skin friction coefficient, \( C_f \), \( \text{Re}_x^{1/2} \), Nusselt number, \( \text{Nu}_x \), \( \text{Re}_x^{-1/2} \), and Sherwood number, \( \text{Sh}_x \), \( \text{Re}_x^{-1/2} \), are demonstrated through figs. 2-19 and tabs. 1(a) and 1(b). The default parameters used throughout the research are \( \lambda = 0.1, \lambda_2 = 0.2, \text{Pr} = 1.0 \) (electrolyte solution), \( Nb = Nt = N = \gamma = 0.3, \alpha_0 = \pi/4, M = Le = 0.7, \) and \( \text{Ec} = 0.2, \) except otherwise mentioned [27, 47]. Precision of the
present codes is guaranteed by comparing the current results with some related existing publications as presented in tab. 1. The comparative study has shown a splendid consistency; thus, the analysis of results can now be proceeded.

Table 1(a). Comparative values of $f'(0)$ for some values of $\lambda_2$ when $\lambda = 0.2, M = \gamma = 0$

<table>
<thead>
<tr>
<th>$\lambda_2$</th>
<th>Exact solution of eq. (12)</th>
<th>[6]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-1.09544512</td>
<td>-1.09641580</td>
<td>-1.09544512</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.92582010</td>
<td>-0.92724220</td>
<td>-0.92582010</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.81649658</td>
<td>-0.81808091</td>
<td>-0.81649658</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.77459667</td>
<td>-0.77618697</td>
<td>-0.77459667</td>
</tr>
</tbody>
</table>

Table 1(b). Comparative values of $\theta'(0)$ for some values of Pr and Ec when $\lambda = \lambda_0 = N_{\theta} = N_{\gamma} = N = L = 0, M = 1.0, \gamma = 0.2, \alpha_2 = \pi/4, Ec = 1.0, and Pr = 1.2$

<table>
<thead>
<tr>
<th>Pr</th>
<th>[31]</th>
<th>Present</th>
<th>Ec</th>
<th>[31]</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.3897</td>
<td>0.38979158</td>
<td>0.0</td>
<td>1.3873</td>
<td>1.38740227</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6219</td>
<td>0.62191866</td>
<td>0.3</td>
<td>1.2125</td>
<td>1.21260213</td>
</tr>
<tr>
<td>1.2</td>
<td>0.8106</td>
<td>0.81061315</td>
<td>0.6</td>
<td>1.0393</td>
<td>1.03934047</td>
</tr>
<tr>
<td>1.5</td>
<td>0.8962</td>
<td>0.89621903</td>
<td>0.9</td>
<td>0.8675</td>
<td>0.86758366</td>
</tr>
</tbody>
</table>

Figures 2-7 depict the changes in the velocity profile $f'(\eta)$ vs. $\eta$ for several parameters such as $\lambda$, $\lambda_2$, $M$, $\gamma$, $N$, and $\alpha_0$, respectively. Figure 2 shows a manifest result that a rise in $\lambda$ tends to decelerate the flow of fluid. The reduction in fluid motion is caused by the domination of relaxation time (time needed for the material to maintain its original position) in comparison to the retardation time (time needed for the material to react to distortion). This has offered more resistance to the fluid-flow and correspondingly, the thickness of momentum boundary-layer is lessened. In relation to the increment of $\lambda_2$ on the velocity profile, the flow of fluid and the thickness of momentum boundary-layer are observed to rise as drawn in fig. 3. Referring to the definition of $\lambda_2$, i. e. $\lambda_2 = \lambda_0 a$, it is patented that $\lambda_2$ and $\lambda_0$ are proportional to one another, where a rise in $\lambda_2$ is strongly influenced by the development in the $\lambda_0$. The
involvement of $\lambda_1$ replicates that the fluid needs to reach an equilibrium state in response to the applied force. Accordingly, the increase in $\lambda_1$ aids in the increment of $\lambda_2$, which then promotes the enhancement of velocity profile. Furthermore, $\lambda_2$ can also measure the elasticity of the materials and highly elastic fluid is associated with higher $\lambda_2$. For small $\lambda_2$, the fluid behaves in a viscous manner and this accounts to the quick convergent of momentum boundary-layer thickness. In fig. 4, the parameter $M$ is investigated to measure the outcome of magnetic field strength on the fluid motion. Physically, higher $M$ implies the stronger magnetic field. That being the case, this will induce the Lorentz force, which is known as the resistive type force or drag force that functions to resist the motion of fluid, so as to decrease the profile for velocity. Figure 5 portrays the influence of several values of $\gamma$ on the velocity profile. From the figure, it is witnessed that the profile for velocity is a growing function of $\gamma$. As a matter of fact, $\gamma$ is depending on the buoyancy force. The increase in $\gamma$ induces the buoyancy force on the external flow, henceforth creates a pleasing pressure gradient that elevates the flow of fluid. Similar trend of graph is noticed in fig. 6, where the impact of $N$ has accelerated the fluid-flow. This situation emerges due to the domination of buoyancy forces over the viscous forces. Buoyancy force has a propensity to speed up the fluid motion, which leads to the augmentation of velocity profile and its associated momentum boundary-layer thickness. Besides, fig. 7 illustrates the opposite behavior of the inclination angle $\alpha_0$, which is in the range of acute angles, i.e., 0 to $\pi/2$. As $\alpha_0$ raises, the fluid-flow is seen to deteriorate. This is due to the decrement of buoyancy ratio strength in the momentum equation, $[\beta_T(T - T_w) + \beta_C(C - C_w)]g \cos \alpha_0$. Evidently, when $\alpha_0 = \pi/2$, the stretched sheet is directed horizontally, and the buoyancy force does no longer exist due to the absence of buoyancy ratio term. Meanwhile, when $\alpha_0 = 0$, the sheet is oriented vertically, and the maximum buoyancy force transpires. Thence, in comparison to the horizontal surface ($\alpha_0 = \pi/2$), it can be deduced that the fluid-flow is higher for the case of vertical surface ($\alpha_0 = 0$).

Figures 8-12 elucidate the impact of the $Ec$, $Nt$ and $Nb$ on the profiles of temperature $\theta(\eta)$ and nanoparticle concentration $\phi(\eta)$, respectively. Clearly, fig. 8 shows that the temperature profile is significantly influenced by Eckert number. In view of physical meaning, Eckert number characterizes the relative contribution of kinetic energy at the wall to the specific enthalpy difference between the wall and the fluid. Escalating the viscous dissipation effect
on fluid-flow enhances the kinetic energy as well as elevates the temperature profile and thermal boundary-layer thickness. Figures 9 and 10 measure the changes of $Nt$ towards the temperature and nanoparticle concentration profiles, in which both profiles are the rising function of $Nt$. The increment is affiliated to the mobile particles near the hot surface which create a thermophoretic force; a force where the motion of nanoparticles is induced by the temperature gradient. It follows that the nanoparticles are blown away from the hot surface of the stretched sheet and a relatively particle-free layer close to the sheet is formed subsequently [5]. On that account, the distribution of nanoparticles becomes strengthen, thereby reinforcing the temperature and nanoparticle concentration profiles. Moreover, it is observed that the graph for temperature is insignificantly varied compared to the nanoparticle concentration. The reason is because $Nt$ is a nanoscale parameter, hence its impact on the temperature is relatively fewer. The consequence of $Nb$ on both temperature and nanoparticle concentration profiles are revealed in figs. 11 and 12. It is crucial to mention that a larger $Nb$ corresponds to higher molecular diffusion with lower viscosity. Increasing $Nb$ will reinforce the collision amongst the
molecules of fluid and its zig zag motion. Such reinforcement has triggered the production of extra heat in the fluid as well as intensified the fluid thermal conductivity, which then prevails upon the increment of temperature and decrement of nanoparticle concentration. In addition, as suggested in many theoretical studies, the Brownian motion of nanoparticles may boost up the thermal conduction through one of the following ways: either the transport of heat on fluid-flow by direct effect of nanoparticles, or an indirect effect of individual nanoparticles attributable to the micro-convection of fluid [48].

Figure 13 is enlisted to study the effect of Lewis number on the nanoparticle concentration profile $\phi(\eta)$ vs. $\eta$. The Lewis number defines the relationship between heat and mass transfer coefficients. Mathematically, Lewis number accounts for the ratio of the Schmidt number to the Prandtl number, that heat and mass will both diffuse at an equal rate when $Le = 1$ and heat will diffuse even more speedily than mass when $Le > 1$. As such, the increase of Lewis number clearly brings forth to the diminution of nanoparticle concentration along with its related concentration boundary-layer thickness.
Figures 14-19 closely describe the variations of the reduced skin friction coefficient, $C_f \sqrt{Re_x}$, Nusselt number, $\frac{1}{2}Nu Re_x^{-1/2}$, and Sherwood number, $\frac{1}{2}Sh Re_x^{-1/2}$, on the diverse values of $\lambda_2, Nt$, and Le. Figure 14 displays the demotion of $C_f \sqrt{Re_x}$ corresponding to the changes occurred in $\lambda_2$ and $Nt$. More specifically, the increase in $\lambda_2$ is to lessen the $C_f \sqrt{Re_x}$. This is physically due to the enhancement of viscoelasticity property of materials that assists in the reduction of the wall shear stress, hence leading to the slackening of the frictional forces. Contrariwise, the increase of $Nt$ has moderately increased the $C_f \sqrt{Re_x}$ and the increment is noticed to be the highest when $\lambda_2 = 0.1$. Since $Nt$ facilitates the nanoparticles diffusion in the boundary-layer, therefore, the distribution of nanoparticles from the heated surface for $\lambda_2 = 0.1$ is manifestly more effective than $\lambda_2 = 1.0$. In figs. 15 and 16, the influences of $\lambda_2$ and $Nt$ have considerably enhanced the $\frac{1}{2}Nu Re_x^{-1/2}$ and $\frac{1}{2}Sh Re_x^{-1/2}$, whereby the results are interrelated with figs. 9 and 10.

These results are foreseeable as the improvement of heat and nanoparticle concentration transfer rates at the surface is predominantly caused by the higher thermal diffusivity offered from the rising $Nt$. As contrasted to fig. 14, it is also found that the heat and nanoparticle concentration transfer rates are highly prominent when $\lambda_2 = 1.0$. Furthermore, when the values of $\lambda_2$ and Lewis number are enhanced, the $C_f \sqrt{Re_x}$ is observed to be decelerating, as shown in fig. 17. In the figure, the presence of parameters $\lambda_2$ and Lewis number has reduced the friction within the fluid and between the fluid and the wall. Figures 18 and 19 demonstrate the rising effects of parameters $\lambda_2$ and Lewis number on the $\frac{1}{2}Nu Re_x^{-1/2}$ and $\frac{1}{2}Sh Re_x^{-1/2}$. The outcome can be precisely linked with fig. 13, wherein the escalation of heat and nanoparticle concentration transfer rates is observed in the consequence of nanoparticle boundary-layer deprivation.

The responses of several continent parameters towards the $C_f \sqrt{Re_x}$, $\frac{1}{2}Nu Re_x^{-1/2}$, and $\frac{1}{2}Sh Re_x^{-1/2}$ are provided in tabs. 2(a) and 2(b). Manifestly, the $C_f \sqrt{Re_x}$ becomes an increasing function of parameters $\lambda_2, \gamma, N, Nt$, and Ec whereas it is a decreasing function for $\lambda_2, M, \alpha, Pr, Nb$, and Le. The $\frac{1}{2}Nu Re_x^{-1/2}$ is a rising function of parameters $\lambda_2, \gamma, N$, and Pr while being a declining function for $\lambda_2, M, \alpha, Nb, Nt, Ec$, and Le. Moreover, the $\lambda_2, \gamma, N, Nb, Ec, Pr$, and Le tend to increase the $\frac{1}{2}Sh Re_x^{-1/2}$ but $\lambda_2, M, \alpha$, and $Nt$ lead to the reduction of $\frac{1}{2}Sh Re_x^{-1/2}$.
Conclusion

The influence of viscous dissipation on MHD mixed convection boundary-layer flow in a Jeffrey fluid induced by an inclined stretching sheet with nanoparticles has been tackled numerically. The flow dynamics of the regime have revealed to be controlled by the parameters \( \lambda, \lambda_2, N, Nb, Nt, Le, \gamma, M, \) and \( \alpha_0 \). As a whole, the vital discoveries of this explorations are itemized as the following:

- Velocity profile is the accelerating function for \( \lambda_2, \gamma, \) and \( N \) while decelerating function for \( \lambda, M, \) and \( \alpha_0 \).
- Temperature profile is the rising function for \( Nt, Nb, \) and \( Ec \).

Table 2(a). Variations of $\frac{1}{2}Re^{1/2}$, $Nu_{Re}^{1/2}$ and $Sh_{Re}^{1/2}$ for diverse values of $\lambda$, $\lambda_2$, $M$, $\gamma$, $N$, and $a_0$ when $Pr = 1.0$, $Nb = Nt = 0.3$, $Ec = 0.2$ and $Le = 0.7$

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<th>$N$</th>
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Table 2(b). Variations of $C_f Re^{1/2}$, $Nu_{Re}^{1/2}$ and $Sh_{Re}^{1/2}$ for diverse values of $Pr$, $Nb$, $Nt$, $Ec$, and $Le$ when $\lambda = 0.1$, $\lambda_2 = 0.2$, $M = 0.7$, $\gamma = 0.3$ and $a_0 = \pi/4$

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- Concentration profile is the accelerating function for $Nt$ while decreasing function for $Nb$ and Lewis number.
- The rise in $\lambda_2$ and Lewis number has worsened the $C_f Re^{1/2}$.
- For the increasing values of $\lambda_2$, $Nt$, and Le, the $Nu_{Re}^{1/2}$ and $Sh_{Re}^{1/2}$ are boosted.

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Nomenclature

- $a, b, d, m, n$ – constants
- $B_0$ – transverse magnetic field, [T]
- $C$ – nanoparticle concentration
- $C_{nf}$ – ambient nanoparticle concentration
- $C_f$ – local skin friction coefficient
- $C_{fr} Re^{1/2}$ – reduced skin friction coefficient
- $C_p$ – specific heat capacity at constant pressure, [Jkg$^{-1}$K$^{-1}$]
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