NEW METHOD FOR SOLVING A CLASS OF FPDE WITH APPLICATIONS

by

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In this work we suggest a numerical approach based on the B-spline polynomial to obtain the solution of linear fractional partial differential equations (FPDE). We find the operational matrix for fractional integration and thenwe convert the main problem into a system of linear algebraic equations by using this matrix. Examples are provided to show the simplicity of our method.

Keywords*: Fractional partial differential equations, Linear B-spline function, Operational matrix, Caputo derivative.*

1. Introduction

Over the last decades fractional calculus were not useful application in physics and mathematical, albeit having a long history. In recent years a number of books [2, 4, 11, 16, 17] on fractional calculus were published.

Compared with orderinary differential equations, fractional order differential equations (FDE) has arbitarery order derivatives and integrals. Our purpose is essentially use the linear B-spline functions to solve PDE in fractional calculus. We attention on the following a class of FPDEs:

$$\frac{\partial^\gamma z}{\partial x^\gamma}+\frac{\partial^\mu z}{\partial t^\mu}=k(x,t), \qquad 0<\mu,\gamma\le 1 \tag{1}$$

s.t

$$\frac{\partial z}{\partial t}|_{x=0}=\psi_1(t), \qquad \frac{\partial z}{\partial x}|_{t=0}=\psi_2(x), \tag{2}$$

$$z(0,t)=\eta_1(t), \qquad z(x,0)=\eta_2(x), \tag{3}$$

where $k,\psi_1,\psi_2,\eta_1,\eta_2$ and $z(x,t)$ are the known functions. $\partial^\gamma z/\partial x^\gamma$ and $\partial^\mu z/\partial t^\mu$ are the fractional Caputo derivative that is given as:

$$\frac{\partial^\gamma z}{\partial t^\gamma}=\begin{cases} I^{m-\gamma}[\dfrac{\partial^m z(x,t)}{\partial x^m}], & \gamma\in(m-1,m),\ \ m\in \mathrm{N}, \\ z^{(m)}(x,t), & \gamma=m \end{cases} \tag{4}$$

Note that

$$(i) \qquad I_x^{\gamma} z(x,t) = \frac{1}{\Gamma(\gamma)} \int_0^x \frac{z(\tau,t)}{(x-\tau)^{1-\gamma}} d\tau, \qquad \gamma > 0, \quad x > 0, \tag{5}$$

$$(ii) \qquad \frac{\partial^{\gamma} x^{\delta}}{\partial x^{\gamma}} = \frac{\Gamma(\delta+1)}{\Gamma(1+\delta-\gamma)} x^{\delta-\gamma}, \qquad \delta > 0, \quad x > 0, \quad \mu > -1, \tag{6}$$

$$(iii) \qquad I_x^{\gamma} \frac{\partial^{\gamma} z(x,t)}{\partial t^{\gamma}} = z(x,t) - \sum_{j=0}^{m-1} \frac{\partial^j z(0^+,t)}{\partial x^j} \frac{x^j}{j!}, \qquad m-1 < \gamma \le m, \tag{7}$$

where I^{γ} is Riemann-Liouville integral operator. There are numerous methods to solve FPDEs. These methods include Adomian decomposition method [5], Fractional sub equation method [9], homotopy perturbation method [7], collocation method [15] , homotopy analysis method [8], He's variational iteration method [6], and other methods [3, 10, 13, 14, 18, 19].

In the current paper, we suggest the linear B-spline operational matrix method to solve the FPDE. At the first, we approximate z in the equation (1) by linear B-spline functions of unknown coeffecients. Then using operational matrixes, the equation (1) convert to a set of algebraic equations. Recently, FPDEs have been solve using Linear B-splines operational matrix of fractional derivatives and B-spline wavelet collocation method [12, 13]. Also Haar wavelet method used to sole these equations [20, 21]. This manuscript is structured as follows: In Section 2, linear B-spline functions on [0,1] and function approximation are explained. Then the operational matrix of fractional integration is computed. In Sections 3, the suggested technique is used to convert FPDEs to algebraic equations . In Section 4, we use the proposed technique to solve some examples. Finally, we conclude with a summary in last Section.

2. B-spline function and Operational matrixes for fractional integration

2.1. Linear B-spline function on [0,1]

The $m$the-order cardinal B-spline $N_m(x)$ has the knote sequence $\{\ldots,-1,0,1,\ldots\}$ Also there are polynomials of order m (degree $m-1$) between the knots. The B-spline functions for $m \ge 2$ on $[0,1]$ has the followin form:

$$N_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (x-k)_+^{m-1}, \tag{8}$$

where $\text{supp}[N_m(x)] = [0,m]$ and for $m=1$ the characteristic function is $N_1(x) = \chi_{[0,1]}(x)$. Here, we apply the linear B-spline function of order 2 in the following form [1]:

$$N_2(x) = \sum_{k=0}^{2} \binom{2}{k} (-1)^k (x-k)_+ = \begin{cases} x, & x \in [0,1), \\ 2-x, & x \in [1,2), \\ 0, & otherwise. \end{cases} \tag{9}$$

Let $N_{j,k}(x) = N_2(2^j x - k), k, j \in Z$ and $B_{j,k}(x) = \text{supp}[N_{j,k}] = close\{x : N_{j,k} \ne 0\}$.
It can show that their support is:

$$B_{j,k} = [2^{-j}k, 2^{-j}(2+k)], \qquad k, j \in Z. \tag{10}$$

Define the set of indices

$$S_j = \{k : B_{j,k} \cap [0,1] \neq 0\}.$$

According to (9) and (10), the minimum and mazumum of $\{S_j\}$ are -1 and $2^j - 1$ Because support of $N_{j,k}$ can be outside of $[0,1]$, so we have to define $\phi_{j,k}$ on $[0,1]$ as follwo:

$$\phi_{j,k} = N_{j,k}(x)\chi_{[0,1]}(x), \qquad j \in Z. \tag{11}$$

2.2. The function approximation

We can expand function $f(x)$ by $\phi_{j,k}$ for a fixed $j = J$ as:

$$f(x) \approx \sum_{k=-1}^{2^J-1} c_k \phi_{J,k}(x) = C^T \Phi_J(x), \tag{12}$$

where the vectors C and Φ_J are

$$C = [c_{-1}, c_0, \ldots, c_{2^{J-1}}]^T, \tag{13}$$

$$\Phi_J(x) = [\phi_{J,-1}(x), \phi_{J,0}(x), \ldots, \phi_{J,2^{J-1}}(x)]^T, \tag{14}$$

with

$$C^T = \left(\int_0^1 f(x)\Phi_J^T(x)dx) \right) P^{-1}, \tag{15}$$

and symmetric matrix is given as:

$$P = \int_0^1 \Phi_J(x)\Phi_J^T(x)dx = \frac{1}{2^{J-2}} \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & & & \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ & & & \frac{1}{24} & \frac{1}{12} \end{bmatrix}_{(2^J+1)\times(2^J+1)} . \tag{16}$$

Also any function $f(x,t)$ could expanded by linear B-spline functions as:

$$f(x,t) \approx \sum_{i=-1}^{2^j-1} \sum_{k=-1}^{2^j-1} f_{ik} \phi_{J,i}(x)\phi_{J,k}(t) \approx \Phi_J^T(x) F \Phi_J(t), \tag{17}$$

where $f_{i,k} = \langle \phi_{J,i}(x), \langle f(x,t), \phi_{J,k}(t) \rangle \rangle$.

2.3. Operational matrix for I^γ

Integration of the vector Φ_J, leads to:

$$I_x^{\gamma}\Phi_J(x) \approx \mathrm{I}^{\gamma}\Phi_J(x), \tag{18}$$

Where I^{γ} is the $(2^J+1)\times(2^J+1)$ operational matrix of fractional integration. We obtain the matrix I^{γ} as follows:

$$\mathrm{I}^{\gamma} = \int_0^1 I_x^{\gamma}\Phi_J(x)\widetilde{\Phi}_J^T(x)dx = \left(\int_0^1 I_x^{\gamma}\Phi_J(x)\Phi_J^T(x)dx\right)P^{-1}, \tag{19}$$

where

$$E = \int_0^1 I_x^{\gamma}\Phi_J(x)\Phi_J^T(x)dx, \tag{20}$$

In Eq. (20), $E=[a_{i,k}]$ is a $(2^J+1)\times(2^J+1)$ matrix as follow:

$$a_{i,k} = \int_0^1 I_x^{\gamma}\phi_{J,k}(x)\phi_{J,i}(x)dx, \qquad i,k=-1\text{K},2^J-1, \tag{21}$$

and $I_x^{\gamma}\phi_{J,k}(x)$ according Eq. (11) can be obtain as:

$$I_x^{\gamma}\phi_{J,k}(x) = I_x^{\gamma}\left(\sum_{i=0}^{2}\binom{2}{i}(-1)^i(2^Jx-(k+i))_+\right)$$

$$= \frac{2^{-J\gamma}}{\Gamma(2+\gamma)}\sum_{i=0}^{2}\binom{2}{i}(-1)^i(2^Jx-(k+i))_+^{2+\gamma}$$

$$= \frac{2^{-J\gamma}}{\Gamma(2+\gamma)}\begin{cases} 0, & x<\frac{k}{2^J}, \\ (2^Jx-k)^{\gamma+1}, & \frac{k}{2^J}\le x<\frac{k+1}{2^J}, \\ (2^Jx-k)^{\gamma+1}-2(2^Jx-(k+1))^{\gamma+1}, & \frac{k+1}{2^J}\le x<\frac{k+2}{2^J}, \\ (2^Jx-k)^{\gamma+1}-2(2^Jx-(k+1))^{\gamma+1}+(2^Jx-(k+2))^{\gamma+1}, & \frac{k+2}{2^J}\le x, \end{cases} \tag{22}$$

So that $I_x^{\gamma}\phi_{J,-1}(x)$ and $I_x^{\gamma}\phi_{J,2^J-1}(x)$ obtain as:

$$I_x^{\gamma}\phi_{J,-1}(x) = \frac{2^{-J\gamma}}{\Gamma(\gamma+1)}\begin{cases} (\gamma+1)(2^Jx)^{\gamma}-(2^Jx)^{\gamma+1}, & x<\frac{1}{2^J}, \\ (\gamma+1)(2^Jx)^{\gamma}-(2^Jx)^{\gamma+1}+(2^Jx-1)^{\gamma+1}, & \frac{1}{2^J}\le x, \end{cases} \tag{23}$$

$$I_x^{\gamma}\phi_{J,2^J-1}(x) = \frac{2^{-J\gamma}}{\Gamma(\gamma+1)}\begin{cases} 0, & x<\frac{2^J-1}{2^J}, \\ (2^Jx-2^J+1)^{\gamma+1}, & \frac{2^J-1}{2^J}\le x<\frac{1}{2^J}, \\ (2^Jx-2^J+1)^{\gamma+1}-(2^Jx-2^J)^{\gamma}(2^Jx-2^J+\gamma+1)^{\gamma+1}, & 1\le x, \end{cases} \tag{24}$$

by substituting $I_x^{\gamma}\phi_{J,k}(x), k=-1,\text{K},2^J-1$ in Eq. (21), we can find matrix E as following:

$$E=\frac{2^{-J(\gamma+1)}}{\Gamma(\gamma+4)}\begin{bmatrix}\kappa_1 & \kappa_2 & \kappa_3 & \cdots & \ldots & \kappa_{2^J} & \kappa_{2^J+1}\\ 1 & \mu_1 & \mu_2 & \cdots & \cdots & \mu_{2^J-1} & \kappa_{2^J}\\ 0 & 1 & \mu_1 & \mu_2 & \cdots & \mu_{2^J-2} & \kappa_{2^J-1}\\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & \ddots & \ddots & \mu_1 & \mu_2 & \kappa_3\\ 0 & 0 & \cdots & 0 & 1 & \mu_1 & \kappa_2\\ 0 & 0 & \cdots & 0 & 0 & 1 & \kappa_1\end{bmatrix}_{(2^J+1)\times(2^J+1)}, \tag{25}$$

where

$\kappa_1=\alpha+2, \quad \kappa_2=(\alpha+1)2^{\alpha+2}-2\alpha-3,$

$\kappa_{i+2}=(i-1)^{\gamma+3}+(\gamma-3(i-1))i^{\gamma+2}-(2\gamma-3(i-1))(i+1)^{\gamma+2}+(\gamma-(i-1))(i+2)^{\gamma+2}, i=1,\ldots,2^J-2$

$k_{2^J+1}=(2^J-2)^{\gamma+3}+2(\gamma-2^J+4)(2^J-1)^{\gamma+2}+((\gamma-2^J+1)^2+3\gamma+5-4.2^J)(2^J)^{\gamma+1},$

$\mu_1=2^{\gamma+3}-4, \quad \mu_2=3^{\gamma+3}-2^{\gamma+5}+6,$

$\kappa_{i+2}=(i+2)^{\gamma+3}-4(i+1)^{\gamma+3}+6i^{\gamma+3}-4(i-1)^{\gamma+3}+(i-2)^{\gamma+3}, i=2,\ldots,2^J-2.$

3. Numerical solution of FPDE

Consider equation (1) with conditions (2) and (3). First, we expand $\frac{\partial^2 z}{\partial x\partial t}\approx \Phi_J^T(x)Z\Phi_J(t)$ then we have:

$$\frac{\partial z}{\partial x}=\int_0^t \frac{\partial^2 z}{\partial x\partial t}dt+\frac{\partial z}{\partial x}|_{t=0}\approx\int_0^t \Phi_J^T(x)Z\Phi_J(t)dt+\psi_2(x)=\Phi_J^T(x)Z\mathrm{I}^1\Phi_J(t)+\psi_2(x), \tag{26}$$

$$\frac{\partial z}{\partial t}=\int_0^x \frac{\partial^2 z}{\partial x\partial t}dx+\frac{\partial z}{\partial x}|_{x=0}\approx\int_0^x \Phi_J^T(x)Z\Phi_J(t)dx+\psi_1(t)=\Phi_J^T(x)[\mathrm{I}^1]^T Z\Phi_J(t)+\psi_1(t). \tag{27}$$

So unknown function $z(x,t)$ obtain as following:

$$z(x,t)\approx\Phi_J^T(x)[\mathrm{I}^1]^T Z\mathrm{I}^1\Phi_J(t)+\int_0^x\psi_2(s)ds+\eta_2(x), \tag{28}$$

Then we have

$$\frac{\partial^\gamma z}{\partial x^\gamma}=I_x^{1-\gamma}\frac{\partial z}{\partial x}\approx I_x^{1-\gamma}(\Phi_J^T(x)Z\mathrm{I}^1\Phi_J(t)+\psi_2(x))=\Phi_J^T(x)[\mathrm{I}^{1-\gamma}]^T Z\mathrm{I}^1\Phi_J(t)+I_x^{1-\gamma}\psi_2(x), \tag{29}$$

$$\frac{\partial^\mu z}{\partial t^\mu}=I_t^{1-\gamma}\frac{\partial z}{\partial t}\approx I_t^{1-\gamma}(\Phi_J^T(x)[\mathrm{I}^1]^T Z\Phi_J(t)+\psi_1(t))=\Phi_J^T(x)[\mathrm{I}^1]^T Z\mathrm{I}^{1-\mu}\Phi_J(t)+I_t^{1-\mu}\psi_1(x), \tag{30}$$

where $g(x,t)=I_x^{1-\gamma}\psi_2(x)+I_t^{1-\mu}\psi_1(t)$ that can be written as follows:

$$g(x,t)\approx\Phi_J^T(x)G\Phi_J(t), \tag{31}$$

and $G=[g_{i,j}]$ is a $(2^J+1)\times(2^J+1)$ matrix. Also we approximate functions $k(x,t)$ by the linear B-spline basis as:

$$k(x,t) \approx \Phi_J^T(x) K \Phi_J(t), \tag{32}$$

Now, by substituting Eqs. (29)-(32) into Eq. (1), we obtain

$$\Phi_J^T(x)[\mathrm{I}^{1-\gamma}]^T Z \mathrm{I}^1 \Phi_J(t) + \Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^{1-\mu} \Phi_J(t) + \Phi_J^T(x) G \Phi_J(t) = \Phi_J^T(x) K \Phi_J(t), \tag{33}$$

or

$$\Phi_J^T(x)([\mathrm{I}^{1-\gamma}]^T Z \mathrm{I}^1 + [\mathrm{I}^1]^T Z \mathrm{I}^{1-\mu} + G - K) \Phi_J(t) = 0. \tag{34}$$

Finally, Eq. (33) give linear system of algebraic equations in the following form:

$$[I^{1-\gamma}]^T Z I^1 + [I^1]^T Z I^{1-\mu} + G - K = 0. \tag{35}$$

So Z can be computed by solving above system. Consequently, we get the numerical solution of $z(x,t)$ using Eq. (28).

4. Numerical examples

Now we solve four examples that shows the efficiency of our technique
Example 1 Analyze the following FPDE [20]:

$$\frac{\partial^{\frac{1}{4}} z}{\partial x^{\frac{1}{4}}} + \frac{\partial^{\frac{1}{4}} z}{\partial t^{\frac{1}{4}}} = \frac{4(x^{\frac{3}{4}} t + x t^{\frac{3}{4}})}{3\Gamma(\frac{3}{4})}, \qquad x,t \in [0,1], \tag{36}$$

subject to

$$\frac{\partial z}{\partial t}|_{x=0} = 0, \qquad \frac{\partial z}{\partial x}|_{t=0} = 0, \tag{37}$$

$$z(0,t) = 0, \qquad z(x,0) = 0, \tag{38}$$

That exact solution is xt.
which is studied byWang et al. [20] by using Haar wavelet. Here we applied the linear B-spline function to solve it. By using Eq. (26) and (27), we have

$$\frac{\partial z}{\partial x} = \int_0^t \frac{\partial^2 z}{\partial x \partial t} dt + \frac{\partial z}{\partial x}|_{t=0} \approx \int_0^t \Phi_J^T(x) Z \Phi_J(t) dt = \Phi_J^T(x) Z \mathrm{I}^1 \Phi_J(t). \tag{39}$$

$$\frac{\partial z}{\partial t} = \int_0^x \frac{\partial^2 z}{\partial x \partial t} dx + \frac{\partial z}{\partial t}|_{x=0} \approx \int_0^x \Phi_J^T(x) Z \Phi_J(t) dx = \Phi_J^T(x)[\mathrm{I}^1]^T Z \Phi_J(t). \tag{40}$$

So unknown function $z(x,t)$ obtain as following:

$$z(x,t) \approx \Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^1 \Phi_J(t) + z(0,t) = \Phi_J^T(x)[\mathrm{I}^1]^T \mathrm{I}^1 Z \Phi_J(t). \tag{41}$$

According Eq. (29) and (30), we have

$$\frac{\partial^{\frac{1}{4}} z}{\partial x^{\frac{1}{4}}} = I_x^{\frac{3}{4}} \frac{\partial z}{\partial x} \approx I_x^{\frac{3}{4}} (\Phi_J^T(x) Z \mathrm{I}^1 \Phi_J(t)) = \Phi_J^T(x) [\mathrm{I}^{\frac{3}{4}}]^T Z \mathrm{I}^1 \Phi_J(t), \tag{42}$$

$$\frac{\partial^{\frac{1}{4}} z}{\partial t^{\frac{1}{4}}} = I_t^{\frac{3}{4}} \frac{\partial z}{\partial t} \approx I_t^{\frac{1}{4}} (\Phi_J^T(x) [\mathrm{I}^1]^T Z \Phi_J(t)) = \Phi_J^T(x) [\mathrm{I}^1]^T Z \mathrm{I}^{\frac{3}{4}} \Phi_J(t), \tag{43}$$

Similarly we approximate $4(x^{3/4}t + xt^{3/4}) / 3\Gamma(3/4)$ as following:

$$\frac{4(x^{\frac{3}{4}}t + xt^{\frac{3}{4}})}{3\Gamma(\frac{3}{4})} \approx \Phi_J^T(x) K \Phi_J(t), \tag{44}$$

and $K = [k_{i,j}]$ is a $(2^J + 1) \times (2^J + 1)$ matrix. Now by substituting Eqs. (42)-(44) into Eq. (36), we have:

$$\Phi_J^T(x) [\mathrm{I}^{\frac{3}{4}}]^T Z \mathrm{I}^1 \Phi_J(t) + \Phi_J^T(x) [\mathrm{I}^1]^T Z \mathrm{I}^{\frac{3}{4}} \Phi_J(t) = \Phi_J^T(x) K \Phi_J(t), \tag{45}$$

or

$$\Phi_J^T(x) ([\mathrm{I}^{\frac{3}{4}}]^T Z I^1 + [\mathrm{I}^1]^T Z \mathrm{I}^{\frac{3}{4}} - K) \Phi_J(t) = 0. \tag{46}$$

Finally, we obtain

$$[\mathrm{I}^{\frac{3}{4}}]^T Z \mathrm{I}^1 + [\mathrm{I}^1]^T Z \mathrm{I}^{\frac{3}{4}} - K = 0. \tag{47}$$

That by solving system (47) we can compute Z.

The numerical results for J = 3 and the exact solutions are plotted in Figs. 1 and 2 respectively. From the Figs. 1, 2 obvious that numerical solutions converge to the exact solution.

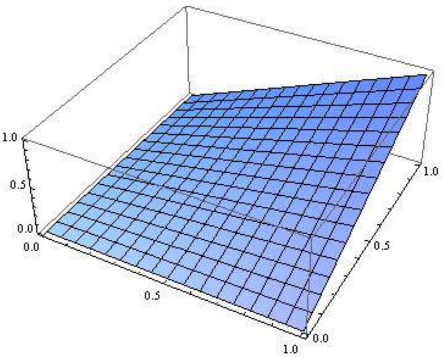

Figure 1: Numerical solution for Example 1 when J = 3.

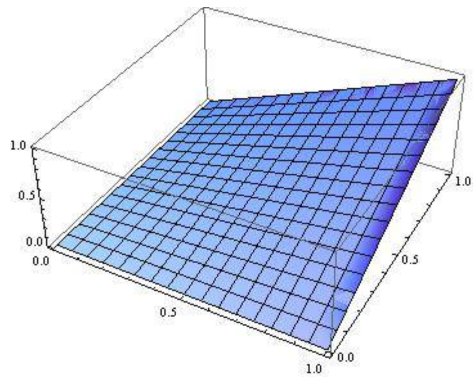

Figure 2: Exact solution for (36).

Example 2 Solve the following fractional PDE [20]

$$\frac{\partial^{\frac{1}{3}} z}{\partial x^{\frac{1}{3}}}+\frac{\partial^{\frac{1}{2}} z}{\partial t^{\frac{1}{2}}}=\frac{\Gamma(3) x^{\frac{5}{3}}}{\Gamma(\frac{8}{3})}+\frac{\Gamma(3) t^{\frac{5}{3}}}{\Gamma(\frac{5}{2})}, \qquad 0 \le x,t \le 1, \tag{48}$$

subject to

$$\frac{\partial z}{\partial t}|_{x=0}=2t, \qquad \frac{\partial z}{\partial x}|_{t=0}=2x, \tag{49}$$

$$z(0,t)=t^2, \qquad z(x,0)=x^2, \tag{50}$$

That exact solution is x^2+t^2.

This example is studied by Wang et al. [20] using Haar wavelet. Here we applied the linear B-spline function to solve it. Let $\partial^2 z/\partial x\partial t \approx \Phi_J^T(x) Z \Phi_J(t)$ then by using Eq. (26) and (27), we have

$$\frac{\partial z}{\partial x}=\int_0^t \frac{\partial^2 z}{\partial x \partial t} dt+\frac{\partial z}{\partial x}|_{t=0} \approx \int_0^t \Phi_J^T(x) Z \Phi_J(t) dt+\frac{\partial z}{\partial x}|_{t=0}=\Phi_J^T(x) Z \mathrm{I}^1 \Phi_J(t)+2x, \tag{51}$$

$$\frac{\partial z}{\partial t}=\int_0^x \frac{\partial^2 z}{\partial x \partial t} dx+\frac{\partial z}{\partial t}|_{x=0} \approx \int_0^x \Phi_J^T(x) Z \Phi_J(t) dx+\frac{\partial z}{\partial t}|_{x=0}=\Phi_J^T(x)[\mathrm{I}^1]^T Z \Phi_J(t)+2t. \tag{52}$$

So unknown function $z(x,t)$ obtain as following:

$$z(x,t) \approx \Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^1 \Phi_J(t)+x^2+z(0,t)=\Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^1 \Phi_J(t)+x^2+t^2. \tag{53}$$

Then

$$\frac{\partial^{\frac{1}{3}} z}{\partial x^{\frac{1}{3}}}=I_x^{\frac{2}{3}} \frac{\partial z}{\partial x} \approx I_x^{\frac{2}{3}}(\Phi_J^T(x) Z \mathrm{I}^1 \Phi_J(t)+2x)=\Phi_J^T(x)[\mathrm{I}^{\frac{2}{3}}]^T Z \mathrm{I}^1 \Phi_J(t)+\frac{2\Gamma(2)}{\Gamma(\frac{8}{3})} x^{\frac{5}{3}}, \tag{54}$$

$$\frac{\partial^{\frac{1}{2}} z}{\partial t^{\frac{1}{2}}}=I_t^{\frac{1}{2}} \frac{\partial z}{\partial t} \approx I_t^{\frac{1}{2}}(\Phi_J^T(x)[\mathrm{I}^1]^T Z \Phi_J(t)+2t)=\Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^{\frac{1}{2}} \Phi_J(t)+\frac{2\Gamma(2)}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}}. \tag{55}$$

Substituting Eqs. (54) and (55) into Eq. (48), we have:

$$\Phi_J^T(x)[\mathrm{I}^{\frac{2}{3}}]^T Z \mathrm{I}^1 \Phi_J(t)+\Phi_J^T(x)[\mathrm{I}^1]^T Z \mathrm{I}^{\frac{1}{2}} \Phi_J(t)=0, \tag{56}$$

Finally, we obtain

$$[\mathrm{I}^{\frac{2}{3}}]^T Z \mathrm{I}^1+[\mathrm{I}^1]^T Z \mathrm{I}^{\frac{1}{2}}=0, \tag{57}$$

So by solving above system we achieve $z=0$. Consequently by subsituting $z=0$ in Eq. (53), we obtain the exact solution of Eq. (48) that is $z(x,t) = x^2 + t^2$.

Example 3 Now we examin the in the FPDE [20]:

$$\frac{\partial^\gamma z}{\partial x^\gamma} + \frac{\partial^\mu z}{\partial t^\mu} = \frac{\Gamma(3)x^{2-\gamma}(t^2+1)}{\Gamma(3-\gamma)} + \frac{\Gamma(3)(x^2+1)t^{2-\mu}}{\Gamma(3-\mu)}, \qquad 0 \le x,t \le 1, \tag{58}$$

subject to

$$\frac{\partial z}{\partial t}|_{x=0} = 2t, \qquad \frac{\partial z}{\partial x}|_{t=0} = 2x, \tag{59}$$

$$z(0,t) = t^2 + 1, \qquad z(x,0) = x^2 + 1, \tag{60}$$

That exact solution of (58) is $(t^2+1)(x^2+1)$.

Fig. 3 and Fig. 4 show the approximation and exact solution of $z(x,t)$ for $J = 3$ when $\gamma = 1/2, \mu = 1/3$ respectively. The numerical results comparing the exact solution for $x = 0.25, J = 3$ are shown in Fig. 5. We can see numerical results converge to exact solution.

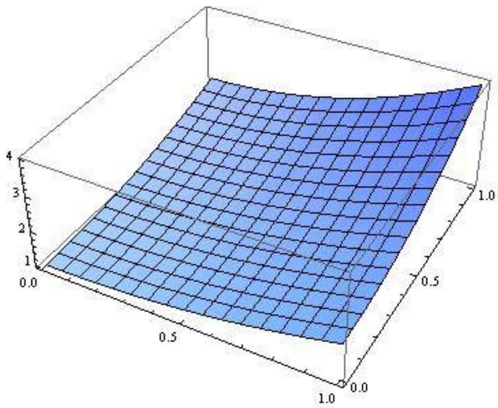

Figure 3: Numerical solution for Example 3 when J = 3.

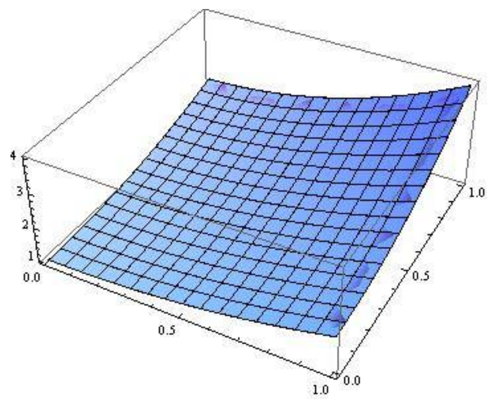

Figure 4: Exact solution for (48).

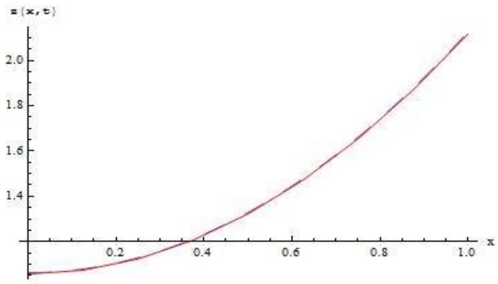

Figure 5: The numerical solution: (dashed) and the exact solution: (Red line) for Example 3 when J = 3.

Example 4 Now we examin the in the FPDE [20]:

$$\frac{\partial^{\gamma} z}{\partial x^{\gamma}} + \frac{\partial^{\mu} z}{\partial t^{\mu}} = \cos x + \cos t, \qquad 0 \le x, t \le 1, \tag{61}$$

subject to

$$\frac{\partial z}{\partial t}|_{x=0} = \cos t, \qquad \frac{\partial z}{\partial x}|_{t=0} = \cos x, \tag{62}$$

$$z(0,t) = \sin t, \qquad z(x,0) = \sin x, \tag{63}$$

That exact solution of this example when $\gamma = \mu = 1$ is $z(x,t) = \sin x + \sin t$.
when $\gamma = \mu = 1$ we obtain $Z = 0$. Consequently, we get the exact solution of Eq. (61) that is $z(x,t) = \sin x + \sin t$. Fig. 6 and Fig. 7 show the numerical solutions for $J = 4$ different values of γ, μ.

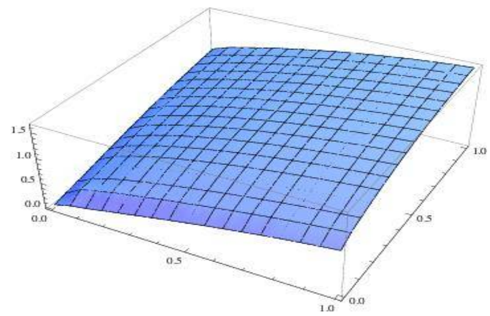

Figure 6: Numerical solution for (61) when $\gamma = 3/4, \mu = 2/3$.

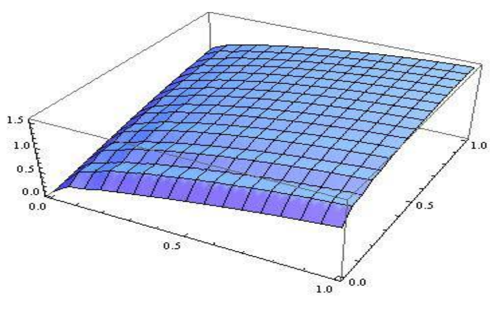

Figure 7: Numerical solution for (61) when $\gamma = 3/5, \mu = 1/3$.

5. Conclusion

In the present paper we used operational matrix of fractional integration based on linear B-spline function to solve the FPDEs. We expand the unknown function with chosen polynomial. The problem has been reduced to a system of algebraic equations . Application examples show good coincidence of the numerical result with exact solution. We used *Mathematica* for computations.

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