NEW METHOD FOR SOLVING A CLASS OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS

by

Hossein JAFARI^{a,b*} and Haleh TAJADODI^c

^a Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa
 ^b Department of Mathematics, University of Mazandaran, Babolsar, Iran
 ^c Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

Original scientific paper https://doi.org/10.2298/TSCI170707031J

In this work we suggest a numerical approach based on the B-spline polynomial to obtain the solution of linear fractional partial differential equations. We find the operational matrix for fractional integration and then we convert the main problem into a system of linear algebraic equations by using this matrix. Examples are provided to show the simplicity of our method.

Key words: fractional partial differential equations, linear B-spline function, operational matrix, Caputo derivative

Introduction

Over the last decades fractional calculus were not useful application in physics and mathematical, albeit having a long history. In recent years a number of books [1-5] on fractional calculus were published.

Compared with ODE, fractional order differential equations (FDE) has arbitrary order derivatives and integrals. Our purpose is essentially useing the linear B-spline functions to solve PDE in fractional calculus. We paid attention on the following a class of fractional PDE (FPDE):

$$\frac{\partial^{\gamma} z}{\partial x^{\gamma}} + \frac{\partial^{\mu} z}{\partial t^{\mu}} = k(x,t), \quad 0 < \mu, \gamma \le 1$$
(1)

s.t

$$\frac{\partial z}{\partial t}\Big|_{x=0} = \psi_1(t), \quad \frac{\partial z}{\partial x}\Big|_{t=0} = \psi_2(x) \tag{2}$$

$$z(0,t) = \eta_1(t), \quad z(x,0) = \eta_2(x) \tag{3}$$

where $k, \psi_1, \psi_2, \eta_1, \eta_2$ are the known function and z(x,t) are the unknown functions. The $\partial^{\gamma} z / \partial x^{\gamma}$ and $\partial^{\mu} z / \partial t^{\mu}$ are the fractional Caputo derivative that is given:

$$\frac{\partial^{\gamma} z(x,t)}{\partial^{\gamma} x} = \begin{cases} I^{m-\gamma} \left[\frac{\partial^{m} z(x,t)}{\partial x^{m}} \right], & \gamma \in (m-1,m), \quad m \in \mathbb{N} \\ z^{(m)}(x,t), & \gamma = m \end{cases}$$
(4)

^{*} Corresponding author, e-mail: jafari.usern@gmail.com

Note that:

$$I_{x}^{\gamma} z(x,t) = \frac{1}{\Gamma(\gamma)} \int_{0}^{x} \frac{z(\tau,t)}{(x-\tau)^{1-\gamma}} d\tau, \quad \gamma > 0, \quad x > 0$$
(5)

$$\frac{\partial^{\gamma} x^{\delta}}{\partial x^{\gamma}} = \frac{\Gamma(\delta+1)}{\Gamma(1+\delta-\gamma)} x^{\delta-\gamma}, \quad \delta > 0, \quad x > 0, \quad \mu > -1$$
(6)

$$I_x^{\gamma} \frac{\partial^{\gamma} z(x,t)}{\partial^{\gamma} x} = z(x,t) - \sum_{j=0}^{m-1} \frac{\partial^j z(0^+,t)}{\partial x^j} \frac{x^j}{j!}, \quad m-1 < \gamma \le m$$
(7)

where I^{γ} is Riemann-Liouville integral operator. There are numerous methods to solve FPDE. These methods include Adomian decomposition method [6], Fractional subequation method [7], homotopy perturbation method [8], collocation method [9], homotopy analysis method [10], He's variational iteration method [11], and other methods [12-17].

In the current paper, we suggest the linear B-spline operational matrix method to solve the FPDE. At the first, we approximate z in the eq. (1) by linear B-spline functions of unknown coefficients. Then using operational matrixes, the eq. (1) convert to a set of algebraic equations. Recently, FPDE have been solveing using Linear B-splines operational matrix of fractional derivatives and B-spline wavelet collocation method [14, 18]. Also Haar wavelet method used to sole these equations [19, 20].

The B-spline function and operational matrixes for fractional integration

Linear B-spline function on [0,1]

The m^{th} order cardinal B-spline $N_m(x)$ has the knote sequence {...,-1,0,1,...} Also there are polynomials of order m (degree m-1) between the knots. The B-spline functions for $m \ge 2$ on [0,1] has the following form:

$$N_m(x) = \frac{1}{(m-1)!} \sum_{k=0}^m \binom{m}{k} (-1)^k (x-k)_+^{m-1}$$
(8)

where supp $[N_m(x)] = [0,m]$ and for m = 1 the characteristic function is $N_1(x) = \chi_{[0,1]}(x)$. Here, we apply the linear B-spline function of order 2 in the following form [21]:

$$N_{2}(x) = \sum_{k=0}^{2} {\binom{2}{k}} (-1)^{k} (x-k)_{+} = \begin{cases} x, & x \in [0,1), \\ 2-x, & x \in [1,2), \\ 0, & \text{otherwise} \end{cases}$$
(9)

Let $N_{j,k}(x) = N_2(2^j x - k), k, j \in \mathbb{Z}$ and $B_{j,k}(x) = \operatorname{supp}[N_{j,k}] = \operatorname{close}\{x : N_{j,k} \neq 0\}$. It can show that their support is:

$$B_{j,k} = [2^{-j}k, 2^{-j}(2+k)], \quad k, j \in \mathbb{Z}$$
(10)

Define the set of indices:

$$S_{i} = \{k : B_{i,k} \cap [0,1] \neq 0\}$$

According to eqs. (9) and (10), the minimum and maximum of $\{S_j\}$ are -1 and $2^j - 1$. Because support of $N_{j,k}$ can be outside of [0,1], so we have to define $\phi_{j,k}$ on [0,1]: Jafari, H., *et al.*: New Method for Solving a Class of Fractional Partial Differential Equations... THERMAL SCIENCE: Year 2018, Vol. 22, Suppl. 1, pp. S277-S286

$$\phi_{j,k} = N_{j,k}(x)\chi_{[0,1]}(x), \quad j \in \mathbb{Z}$$
(11)

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The function approximation

We can expand function f(x) by $\phi_{j,k}$ for a fixed j = J:

$$f(x) \approx \sum_{k=-1}^{2^{J}-1} c_{k} \phi_{J,k}(x) = C^{T} \Phi_{J}(x)$$
(12)

where the vectors C and Φ_J are:

$$C = [c_{-1}, c_0, \dots, c_{2^{J-1}}]^T$$
(13)

$$\Phi_{J}(x) = \left[\phi_{J,-1}(x), \phi_{J,0}(x), \dots, \phi_{J,2^{J-1}}(x)\right]^{T}$$
(14)

with

$$C^{T} = \left[\int_{0}^{1} f(x)\Phi_{J}^{T}(x)\mathrm{d}x\right]P^{-1}$$
(15)

and symmetric matrix is given:

$$P = \int_{0}^{1} \Phi_{J}(x) \Phi_{J}^{T}(x) dx = \frac{1}{2^{J-2}} \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & & \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & \\ & \ddots & \ddots & \ddots \\ & & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ & & & \frac{1}{24} & \frac{1}{12} \end{bmatrix}_{(2^{J}+1) \times (2^{J}+1)}$$
(16)

Also any function f(x,t) could expand by linear B-spline functions:

$$f(x,t) \approx \sum_{i=-1}^{2^{J}-1} \sum_{k=-1}^{2^{J}-1} f_{ik} \phi_{J,i}(x) \phi_{J,k}(t) \approx \Phi_{J}^{T}(x) F \Phi_{J}(t)$$
(17)

where $f_{i,k} = \langle \phi_{J,i}(x), \langle f(x,t), \phi_{J,k}(t) \rangle \rangle$.

Operational matrix for I^{γ}

Integration of the vector Φ_J , leads to:

$$I_{j}^{\gamma} \Phi_{J}(x) \approx I^{\gamma} \Phi_{J}(x), \qquad (18)$$

where I^{γ} is the $(2^{J} + 1) \times (2^{J} + 1)$ operational matrix of fractional integration. We obtain the matrix I^{γ} :

$$I^{\gamma} = \int_{0}^{1} I_{x}^{\gamma} \Phi_{J}(x) \tilde{\Phi}_{J}^{T}(x) dx = \left[\int_{0}^{1} I_{x}^{\gamma} \Phi_{J}(x) \Phi_{J}^{T}(x) dx\right] P^{-1}$$
(19)

where

$$E = \int_{0}^{1} I_x^{\gamma} \Phi_J(x) \Phi_J^T(x) \mathrm{d}x$$
(20)

In eq. (20), $E = [a_{i,k}]$ is a $(2^J + 1) \times (2^J + 1)$ matrix:

$$a_{i,k} = \int_{0}^{1} I_{x}^{\gamma} \phi_{J,k}(x) \phi_{J,i}(x) dx, \quad i,k = -1..., 2^{J} - 1$$
(21)

and $I_x^{\gamma} \phi_{J,k}(x)$ according eq. (11) can be obtain:

$$I_{x}^{\gamma}\phi_{J,k}(x) = I_{x}^{\gamma}\left\{\sum_{i=0}^{2}\binom{2}{i}(-1)^{i}\left[2^{J}x - (k+i)\right]_{+}\right\} =$$

$$= \frac{2^{-J\gamma}}{\Gamma(2+\gamma)}\sum_{i=0}^{2}\binom{2}{i}(-1)^{i}\left[2^{J}x - (k+i)\right]_{+}^{1+\gamma} =$$

$$\left\{0, \quad x < \frac{k}{2^{J}}, \\ (2^{J}x - k)^{\gamma+1}, \quad \frac{k}{2^{J}} \le x < \frac{k+1}{2^{J}}, \\ (2^{J}x - k)^{\gamma+1} - 2\left[2^{J}x - (k+1)\right]^{\gamma+1}, \quad \frac{k+1}{2^{J}} \le x < \frac{k+2}{2^{J}} \\ (2^{J}x - k)^{\gamma+1} - 2\left[2^{J}x - (k+1)\right]^{\gamma+1} + \left[2^{J}x - (k+2)\right]^{\gamma+1} - \frac{k+2}{2} \le x \right\}$$

So that
$$I_x^{\gamma} \phi_{J,-1}(x)$$
 and $I_x^{\gamma} \phi_{J,2^{J}-1}(x)$ obtain:

$$\frac{\left[(2^J x - k)^{\gamma+1} - 2\left[2^J x - (k+1)\right]^{\gamma+1} + \left[2^J x - (k+2)\right]^{\gamma+1}, \quad \frac{k+2}{2^J} \le x$$

$$I_{x}^{\gamma}\phi_{J,-1}(x) = \frac{2^{-J\gamma}}{\Gamma(\gamma+2)} \begin{cases} (\gamma+1)(2^{J}x)^{\gamma} - (2^{J}x)^{\gamma+1}, & x < \frac{1}{2^{J}} \\ (\gamma+1)(2^{J}x)^{\gamma} - (2^{J}x)^{\gamma+1} + (2^{J}x-1)^{\gamma+1}, & \frac{1}{2^{J}} \le x \end{cases}$$
(23)

$$I_{x}^{\gamma}\phi_{J,2^{J}-1}(x) = \frac{2^{-J\gamma}}{\Gamma(\gamma+2)} \begin{cases} 0, & x < \frac{2^{J}-1}{2^{J}} \\ (2^{J}x-2^{J}+1)^{\gamma+1}, & \frac{2^{J}-1}{2^{J}} \le x < \frac{1}{2^{J}} \\ (2^{J}x-2^{J}+1)^{\gamma+1} - (2^{J}x-2^{J})^{\gamma} (2^{J}x-2^{J}+\gamma+1)^{\gamma+1}, & 1 \le x \end{cases}$$
(24)

by substituting $I_x^{\gamma} \varphi_{J,k}(x), k = -1, \dots, 2^J - 1$ in eq. (21), we can find matrix E:

$$E = \frac{2^{-J(\gamma+1)}}{\Gamma(\gamma+4)} \begin{bmatrix} \kappa_1 & \kappa_2 & \kappa_3 & \cdots & \cdots & \kappa_{2^J} & \kappa_{2^{J+1}} \\ 1 & \mu_1 & \mu_2 & \cdots & \mu_{2^{J}-1} & \kappa_{2^J} \\ 0 & 1 & \mu_1 & \mu_2 & \cdots & \mu_{2^{J}-2} & \kappa_{2^{J}-1} \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \mu_1 & \kappa_2 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \kappa_1 \end{bmatrix}_{(2^J+1)\times(2^J+1)}$$
(25)

where

$$\begin{split} \kappa_{1} &= \gamma + 2, \quad \kappa_{2} = (\gamma + 1)2^{\gamma + 2} - 2\gamma - 3\\ \kappa_{i+2} &= (i-1)^{\gamma + 3} + \left[\gamma - 3(i-1)\right]i^{\gamma + 2} - \left[2\gamma - 3(i-1)\right](i+1)^{\gamma + 2} + \\ &+ \left[\gamma - (i-1)\right](i+2)^{\gamma + 2}, \quad i = 1, \dots, 2^{J} - 2\\ k_{2^{J} + 1} &= (2^{J} - 2)^{\gamma + 3} + 2(\gamma - 2^{J} + 4)(2^{J} - 1)^{\gamma + 2} + \left[(\gamma - 2^{J} + 1)^{2} + 3\gamma + 5 - 4.2^{J}\right](2^{J})^{\gamma + 1}\\ \mu_{1} &= 2^{\gamma + 3} - 4, \quad \mu_{2} = 3^{\gamma + 3} - 2^{\gamma + 5} + 6\\ \kappa_{i+2} &= (i+2)^{\gamma + 3} - 4(i+1)^{\gamma + 3} + 6i^{\gamma + 3} - 4(i-1)^{\gamma + 3} + (i-2)^{\gamma + 3}, \quad i = 2, \dots, 2^{J} - 2 \end{split}$$

Numerical solution of FPDE

Consider eq. (1) with conditions (2) and (3). First, we expand

$$\frac{\partial^2 z}{\partial x \partial t} \approx \Phi_J^T(x) Z \Phi_J(t)$$

then we have:

$$\frac{\partial z}{\partial x} = \int_{0}^{t} \frac{\partial^{2} z}{\partial x \partial t} dt + \frac{\partial z}{\partial x} \Big|_{t=0} \approx \int_{0}^{t} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dt + \psi_{2}(x) = \Phi_{J}^{T}(x) Z I^{1} \Phi_{J}(t) + \psi_{2}(x)$$
(26)

$$\frac{\partial z}{\partial t} = \int_{0}^{x} \frac{\partial^2 z}{\partial x \partial t} dx + \frac{\partial z}{\partial x} \Big|_{x=0} \approx \int_{0}^{x} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dx + \psi_{1}(t) = \Phi_{J}^{T}(x) [I^{1}]^{T} Z \Phi_{J}(t) + \psi_{1}(t)$$
(27)

So unknown function z(x,t) obtain:

$$z(x,t) \approx \Phi_J^T(x) [I^1]^T Z I^1 \Phi_J(t) + \int_0^x \psi_2(s) ds + \eta_2(x)$$
(28)

Then we have:

$$\frac{\partial^{\gamma} z}{\partial x^{\gamma}} = I_x^{1-\gamma} \frac{\partial z}{\partial x} \approx I_x^{1-\gamma} \Big[\Phi_J^T(x) Z I^1 \Phi_J(t) + \psi_2(x) \Big] = \Phi_J^T(x) [I^{1-\gamma}]^T Z I^1 \Phi_J(t) + I_x^{1-\gamma} \psi_2(x)$$
(29)

$$\frac{\partial^{\mu}z}{\partial t^{\mu}} = I_{t}^{1-\mu} \frac{\partial z}{\partial t} \approx I_{t}^{1-\mu} \Big[\Phi_{J}^{T}(x) [I^{1}]^{T} Z \Phi_{J}(t) + \psi_{1}(t) \Big] = \Phi_{J}^{T}(x) [I^{1}]^{T} Z I^{1-\mu} \Phi_{J}(t) + I_{t}^{1-\mu} \psi_{1}(x)$$
(30)

where $g(x,t) = I_x^{1-\gamma} \psi_2(x) + I_t^{1-\mu} \psi_1(t)$ that can be written:

$$g(x,t) \approx \Phi_J^T(x) G \Phi_J(t) \tag{31}$$

and $G = [g_{i,j}]$ is a $(2^{J} + 1) \times (2^{J} + 1)$ matrix. Also we approximate functions k(x,t) by the linear B-spline basis:

$$k(x,t) \approx \Phi_J^T(x) K \Phi_J(t) \tag{32}$$

Now, by substituting eqs. (29)-(32) into eq. (1), we obtain:

$$\Phi_{J}^{T}(x)[I^{1-\gamma}]^{T}ZI^{1}\Phi_{J}(t) + \Phi_{J}^{T}(x)[I^{1}]^{T}ZI^{1-\mu}\Phi_{J}(t) + \Phi_{J}^{T}(x)G\Phi_{J}(t) = \Phi_{J}^{T}(x)K\Phi_{J}(t)$$
(33)

or

$$\Phi_{J}^{T}(x)\left\{ \left[I^{1-\gamma}\right]^{T}ZI^{1} + \left[I^{1}\right]^{T}ZI^{1-\mu} + G - K \right\} \Phi_{J}(t) = 0$$
(34)

Finally, eq. (33) give linear system of algebraic equations in the following form:

$$[I^{1-\gamma}]^{T}ZI^{1} + [I^{1}]^{T}ZI^{1-\mu} + G - K = 0$$
(35)

So Z can be computed by solving previous system. Consequently, we get the numerical solution of z(x,t) using eq. (28).

Numerical examples

Now we solve four examples that shows the efficiency of our technique. *Example 1.* Analyze the following FPDE [19]:

$$\frac{\partial^{1/4}z}{\partial x^{1/4}} + \frac{\partial^{1/4}z}{\partial t^{1/4}} = \frac{4(x^{3/4}t + xt^{3/4})}{3\Gamma(3/4)}, \quad x, t \in [0, 1]$$
(36)

subject to:

$$\frac{\partial z}{\partial t}\Big|_{x=0} = 0, \quad \frac{\partial z}{\partial x}\Big|_{t=0} = 0 \tag{37}$$

$$z(0,t) = 0, \quad z(x,0) = 0$$
 (38)

That exact solution is xt which is studied by Wang *et al.* [19] by using Haar wavelet. Here we applied the linear B-spline function to solve it. By using eqs. (26) and (27), we have:

$$\frac{\partial z}{\partial x} = \int_{0}^{t} \frac{\partial^2 z}{\partial x \partial t} dt + \frac{\partial z}{\partial x} \Big|_{t=0} \approx \int_{0}^{t} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dt = \Phi_{J}^{T}(x) Z I^{1} \Phi_{J}(t)$$
(39)

$$\frac{\partial z}{\partial t} = \int_{0}^{x} \frac{\partial^{2} z}{\partial x \partial t} dx + \frac{\partial z}{\partial t} \Big|_{x=0} \approx \int_{0}^{x} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dx = \Phi_{J}^{T}(x) [I^{1}]^{T} Z \Phi_{J}(t)$$
(40)

So unknown function z(x,t) obtain:

$$z(x,t) \approx \Phi_J^T(x) [I^1]^T Z I^1 \Phi_J(t) + z(0,t) = \Phi_J^T(x) [I^1]^T I^1 Z \Phi_J(t)$$
(41)

According eqs. (29) and (30), we have:

$$\frac{\partial^{1/4}z}{\partial x^{1/4}} = I_x^{3/4} \frac{\partial z}{\partial x} \approx I_x^{3/4} [\Phi_J^T(x) Z I^1 \Phi_J(t)] = \Phi_J^T(x) [I^{3/4}]^T Z I^1 \Phi_J(t)$$
(42)

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 $\frac{\partial^{1/4} z}{\partial t^{1/4}} = I_t^{3/4} \frac{\partial z}{\partial t} \approx I_t^{1/4} \{ \Phi_J^T(x) [\mathbf{I}^1]^T Z \Phi_J(t) \} = \Phi_J^T(x) [\mathbf{I}^1]^T Z \mathbf{I}^{3/4} \Phi_J(t)$ (43)

Similarly we approximate $4(x^{3/4}t + xt^{3/4})/3\Gamma(3/4)$:

$$\frac{4(x^{3/4}t + xt^{3/4})}{3\Gamma(3/4)} \approx \Phi_J^T(x)K\Phi_J(t)$$
(44)

and $K = [k_{i,j}]$ is a $(2^{j} + 1) \times (2^{j} + 1)$ matrix. Now by substituting eqs. (42)-(44) into eq. (36), we have:

$$\Phi_{J}^{T}(x)[I^{3/4}]^{T}ZI^{1}\Phi_{J}(t) + \Phi_{J}^{T}(x)[I^{1}]^{T}ZI^{3/4}\Phi_{J}(t) = \Phi_{J}^{T}(x)K\Phi_{J}(t), \qquad (45)$$

or

$$\Phi_J^T(x)\{[\mathbf{I}^{3/4}]^T Z I^1 + [\mathbf{I}^1]^T Z \mathbf{I}^{3/4} - K\} \Phi_J(t) = 0$$
(46)

Finally, we obtain:

$$[\mathbf{I}^{3/4}]^T Z \mathbf{I}^1 + [\mathbf{I}^1]^T Z \mathbf{I}^{3/4} - K = 0$$
(47)

That by solving system eq. (47) we can compute Z.

The numerical results for J = 3 and the exact solutions are plotted in figs. 1 and 2, respectively. From figs. 1 and 2 is obvious that numerical solutions converge to the exact solution.



example 1 when J = 3(for color image see journal web site)

(for color image see journal web site)

Example 2. Solve the following fractional PDE [19]:

$$\frac{\partial^{1/3} z}{\partial x^{1/3}} + \frac{\partial^{1/2} z}{\partial t^{1/2}} = \frac{\Gamma(3) x^{5/3}}{\Gamma(8/3)} + \frac{\Gamma(3) t^{5/3}}{\Gamma(5/2)}, \quad 0 \le x, \quad t \le 1$$
(48)

subject to:

$$\frac{\partial z}{\partial t}\Big|_{x=0} = 2t, \quad \frac{\partial z}{\partial x}\Big|_{t=0} = 2x \tag{49}$$

$$z(0,t) = t^2, \quad z(x,0) = x^2$$
 (50)

That exact solution is $x^2 + t^2$.

This example is studied by Wang *et al.* [19] using Haar wavelet. Here we applied the linear B-spline function to solve it. Let $\partial^2 z / \partial x \partial t \approx \Phi_J^T(x) Z \Phi_J(t)$ then by using eqs. (26) and (27), we have:

$$\frac{\partial z}{\partial x} = \int_{0}^{t} \frac{\partial^{2} z}{\partial x \partial t} dt + \frac{\partial z}{\partial x} \Big|_{t=0} \approx \int_{0}^{t} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dt + \frac{\partial z}{\partial x} \Big|_{t=0} = \Phi_{J}^{T}(x) Z I^{1} \Phi_{J}(t) + 2x$$
(51)

$$\frac{\partial z}{\partial t} = \int_{0}^{x} \frac{\partial^{2} z}{\partial x \partial t} dx + \frac{\partial z}{\partial t} \Big|_{x=0} \approx \int_{0}^{x} \Phi_{J}^{T}(x) Z \Phi_{J}(t) dx + \frac{\partial z}{\partial t} \Big|_{x=0} = \Phi_{J}^{T}(x) [I^{1}]^{T} Z \Phi_{J}(t) + 2t$$
(52)

So unknown function z(x,t) obtain:

$$z(x,t) \approx \Phi_J^T(x) [I^1]^T Z I^1 \Phi_J(t) + x^2 + z(0,t) = \Phi_J^T(x) [I^1]^T Z I^1 \Phi_J(t) + x^2 + t^2$$
(53)

Then

$$\frac{\partial^{1/3} z}{\partial x^{1/3}} = I_x^{2/3} \frac{\partial z}{\partial x} \approx I_x^{2/3} [\Phi_J^T(x) Z I^1 \Phi_J(t) + 2x] = \Phi_J^T(x) [I^{2/3}]^T Z I^1 \Phi_J(t) + \frac{2\Gamma(2)}{\Gamma(8/3)} x^{5/3}$$
(54)

$$\frac{\partial^{1/2} z}{\partial t^{1/2}} = I_t^{1/2} \frac{\partial z}{\partial t} \approx I_t^{1/2} \{ \Phi_J^T(x) [I^1]^T Z \Phi_J(t) + 2t \} = \Phi_J^T(x) [I^1]^T Z I^{1/2} \Phi_J(t) + \frac{2\Gamma(2)}{\Gamma(5/2)} t^{3/2}$$
(55)

Substituting eqs. (54) and (55) into eq. (48), we have:

$$\Phi_J^T(x)[\mathbf{I}^{2/3}]^T Z \mathbf{I}^1 \Phi_J(t) + \Phi_J^T(x)[\mathbf{I}^1]^T Z \mathbf{I}^{1/2} \Phi_J(t) = 0$$
(56)

Finally, we obtain:

$$[I^{2/3}]^T Z I^1 + [I^1]^T Z I^{1/2} = 0$$
(57)

So by solving previous system we achieve z = 0. Consequently by substituting z = 0 in eq. (53), we obtain the exact solution of eq. (48) that is $z(x,t) = x^2 + t^2$.

Example 3. Now we examine the numerical solution of the FPDE [19]:

$$\frac{\partial^{\gamma} z}{\partial x^{\gamma}} + \frac{\partial^{\mu} z}{\partial t^{\mu}} = \frac{\Gamma(3)x^{2-\gamma}(t^2+1)}{\Gamma(3-\gamma)} + \frac{\Gamma(3)(x^2+1)t^{2-\mu}}{\Gamma(3-\mu)}, \quad 0 \le x, \quad t \le 1$$
(58)

subject to:

$$\frac{\partial z}{\partial t}|_{x=0} = 2t, \quad \frac{\partial z}{\partial x}|_{t=0} = 2x \tag{59}$$

$$z(0,t) = t^{2} + 1, \quad z(x,0) = x^{2} + 1$$
 (60)

That exact solution of eq. (58) is $(t^2 + 1)(x^2 + 1)$.

Figures 3 and 4 show the approximation and exact solution of z(x,t) for J = 3 when $\gamma = 1/2$, $\mu = 1/3$, respectively. The numerical results comparing the exact solution for x = 0.25, J = 3 are shown in fig. 5. We can see numerical results converge to exact solution. *Example 4.* Now we examine the numerical solution of the FPDE [19]:

$$\frac{\partial^{\gamma} z}{\partial x^{\gamma}} + \frac{\partial^{\mu} z}{\partial t^{\mu}} = \cos x + \cos t, \quad 0 \le x, \ t \le 1$$
(61)

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Figure 5. The numerical solution: (dashed) and the exact solution: (Red line) for *example 3* when J = 3(for color image see journal web site)

image see journal web site)



subject to:

$$\frac{\partial z}{\partial t}|_{x=0} = \cos t, \quad \frac{\partial z}{\partial x}|_{t=0} = \cos x \tag{62}$$

$$z(0,t) = \sin t, \quad z(x,0) = \sin x$$
 (63)

That exact solution of this example when $\gamma = \mu = 1$ is $z(x,t) = \sin x + \sin t$. When $\gamma = \mu = 1$ we obtain Z = 0. Consequently, we get the exact solution of eq. (61) that is $z(x,t) = \sin x + \sin t$. Figures 6 and 7 show the numerical solutions for J = 4 different values of γ and μ .



(61) when $\gamma = 3/4, \mu = 2/3$ (for color when $\gamma = 3/5$, $\mu = 1/3$ (for color image see journal web site)

Conclusion

In the present paper we used operational matrix of fractional integration based on linear B-spline function to solve the FPDE. We expand the unknown function with chosen polynomial. The problem has been reduced to a system of algebraic equations. Application examples show good coincidence of the numerical result with exact solution. We used Mathematica for computations.

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