ON NABLA DISCRETE FRACTIONAL CALCULUS OPERATOR FOR A MODIFIED BESSEL EQUATION

by

Resat YILMAZER[a*] and Okkes OZTURK[b]

[a] Department of Mathematics, Firat University, Elazig, Turkey
[b] Department of Mathematics, Bitlis Eren University, Bitlis, Turkey

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In thermal sciences, it is possible to encounter topics such as Bessel beams, Bessel functions or Bessel equations. In this work, we also present new discrete fractional solutions of the modified Bessel differential equation by means of the nabla-discrete fractional calculus operator. We consider homogeneous and non-homogeneous modified Bessel differential equation. So, we acquire four new solutions of these equations in the discrete fractional forms via a newly developed method.

Key words: *discrete fractional calculus, modified Bessel equation, nabla operator*

Introduction

The fractional calculus studies that started about 300 years ago have been an interesting subject of present day. This theory, which spans a wide field, has contributed many science fields and has been gained a lot of scientific publications to the literature [1-5]. Recently, there appeared a number of papers on the discrete fractional calculus, which has helped to build up some of the basic theory of this area. For example, Atici and Eloe [6] introduced the discrete Laplace transform method for a family of finite fractional difference equations. Atici and Eloe [7] defined the initial value problems in the discrete fractional calculus. Atici and Eloe [8] studied properties of discrete fractional calculus with the nabla operator. They developed exponential laws and the product rule to the forward fractional calculus. Atici and Sengul [9] developed the Leibniz rule and summation by parts formula in discrete fractional calculus. Bastos and Torres [10] presented the more general discrete fractional operator and this operator was defined by the delta and nabla fractional sums. Holm [11] introduced fractional sums and difference operators. Holm applied this theory to solve the fractional initial value problems. Anastassiou [12] derived the right discrete nabla fractional Taylor formula. Mohan [13] discussed the differentiability properties of solutions of nabla fractional difference equations of order non-integer. Han *et al.* [14] studied existence and non-existence of positive solutions to the discrete fractional boundary value problems.

Solutions of the fractional modified Bessel equation were obtained and, asymptotic analysis was provided for these solutions in [15]. Robin [16] studied on Bessel equations and Bessel functions via an elementary factorization method. The solutions of the classical Bessel equation were presented by means of the fractional calculus theorems [17]. Before giving some results concerning the Bessel equation, we should give its physical properties. The total energy

* Corresponding author, e-mail: rstyilmazer@gmail.com

of the particle is given by $E = p^2/2M = \hbar^2 k^2/2M = k^2$ where p is its initial or final momentum, and k the corresponding wavenumber [18]. Bessel beams, Bessel functions or Bessel equations can be seen in thermal sciences. For example, Doan *et al.* [19] presented a short paper about Bessel beam laser-scribing of thin film silicon solar cells. Semiclassical dynamics of dilute thermal atom clouds located in 3-D optical lattices generated by stationary optical Bessel beams were characterized [20]. A method to generate a Bessel beam using cross-phase modulation based on the thermal non-linear optical effect was introduced [21]. The expansions of the modified Bessel functions were used to obtain the temperature field [22]. Jones [23] took advantage of the Bessel functions in his book on thermal sciences. The length of the temperature plume was calculated via an approximation of the modified Bessel function [24]. Swain *et al.* [25] were studied on a straight triangular fin and a general porous pin fin profile. To formulate heat transfer equation in straight triangular fin modified Bessel's equation was used.

The aim of present study to solve the homogeneous and non-homogeneous modified Bessel equation by using the nabla-discrete fractional calculus operator.

Preliminaries

Here, we present some essential information about discrete fractional calculus theory. We use the some notations: $\mathbb{N}$ is the set of natural numbers including zero and $\mathbb{Z}$ is the set of integers. The $\mathbb{N}_b = \{b, b+1, b+2, \cdots\}$ for $b \in \mathbb{Z}$. Let $f(t)$ and $g(t)$ be a real valued function defined on $\mathbb{N}_0^+$. These and other related results can be found in [6-14, 26-29].

Definition 1. The rising factorial power is given:

$$t^{\overline{n}} = t(t+1)(t+2)\ldots(t+n-1), \quad n \in \mathbb{N}, \quad t^{\overline{0}} = 1$$

Let α a real number. Then $t^{\overline{\alpha}}$ is defined:

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)} \tag{1}$$

where $t \in \mathbb{R} \backslash \{\ldots, -2, -1, 0\}$ and $0^{\overline{\alpha}} = 0$. Let us note that:

$$\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}} \tag{2}$$

where $\nabla u(t) = u(t) - u(t-1)$. For n = 2, 3, ... define ∇^n indeductively by $\nabla^n = \nabla\nabla^{n-1}$.

Definition 2. The α^{th} order fractional sum of f is given:

$$\nabla_b^{-\alpha} f(t) = \sum_{s=b}^{t} \frac{[t-\delta(t)]^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s) \tag{3}$$

where $t \in \mathbb{N}_b$, $\delta(t) = t-1$ is backward jump operator of the time scale calculus.

Theorem 3. Let $f(t)$ and $g(t)$: $\mathbb{N}_0^+ \to \mathbb{R}, \alpha, \beta > 0$ and h, v are scalars. The following equality holds:

$$\nabla^{-\alpha}\nabla^{-\beta} f(t) = \nabla^{-(\alpha+\beta)} f(t) = \nabla^{-\beta}\nabla^{-\alpha} f(t) \tag{4}$$

$$\nabla^{\alpha}\left[hf(t) + vg(t)\right] = h\nabla^{\alpha} f(t) + v\nabla^{\alpha} g(t) \tag{5}$$

$$\nabla\nabla^{-\alpha} f(t) = \nabla^{-(\alpha-1)} f(t) \tag{6}$$

$$\nabla^{-\alpha}\nabla f(t)=\nabla^{(1-\alpha)}f(t)-\binom{t+\alpha-2}{t-1}f(0) \tag{7}$$

Lemma 4. For any $\alpha > 0$, α^{th} order fractional difference of the product fg is given:

$$\nabla_0^{\alpha}(fg)(t)=\sum_{n=0}^{t}\binom{\alpha}{n}\left[\nabla_0^{\alpha-n}f(t-n)\right]\left[\nabla^{n}g(t)\right] \tag{8}$$

Lemma 5. If the function $f(t)$ is single valued and analytic, then:

$$\left[f_\alpha(t)\right]_\beta=f_{\alpha+\beta}(t)=\left[f_\beta(t)\right]_\alpha \quad \left[f_\alpha(t)\neq 0, f_\beta(t)\neq 0, \alpha, \beta\in\mathbb{R}, t\in\mathbb{N}\right]$$

Main results

Here, we will present the discrete fractional solutions of the modified Bessel differential equation by applying the nabla-discrete fractional calculus operator by way of the following theorems.

Theorem 6. Let $\psi\in\{\psi: 0\neq|\psi_\alpha|<\infty, \alpha\in\mathbb{R}\}$ and $f\in\{f: 0\neq|f_\alpha|<\infty, \alpha\in\mathbb{R}\}$. Then the non-homogeneous modified Bessel equation:

$$\psi_2+\psi\left(-k^2+\frac{1/4-\nu^2}{r^2}\right)=f \quad (r\neq 0) \tag{9}$$

has particular solutions of the forms:

$$\psi^{I}=r^{\nu+1/2}\mathrm{e}^{kr}\left\{\left[\left(fr^{-\nu+1/2}\mathrm{e}^{-kr}\right)_{-q^{-1}(\nu+1/2)}r^{\nu-1/2}\mathrm{e}^{2kr}\right]_{-1}r^{-(\nu+1/2)}\mathrm{e}^{-2kr}\right\}_{-1+q^{-1}(\nu+1/2)} \tag{10}$$

$$\psi^{II}=r^{\nu+1/2}\mathrm{e}^{-kr}\left\{\left[\left(fr^{-\nu+1/2}\mathrm{e}^{kr}\right)_{-q^{-1}(\nu+1/2)}r^{\nu-1/2}\mathrm{e}^{-2kr}\right]_{-1}r^{-(\nu+1/2)}\mathrm{e}^{2kr}\right\}_{-1+q^{-1}(\nu+1/2)} \tag{11}$$

$$\psi^{III}=r^{-\nu+1/2}\mathrm{e}^{kr}\left\{\left[\left(fr^{\nu+1/2}\mathrm{e}^{-kr}\right)_{q^{-1}(\nu-1/2)}r^{-(\nu+1/2)}\mathrm{e}^{2kr}\right]_{-1}r^{\nu-1/2}\mathrm{e}^{-2kr}\right\}_{-1+q^{-1}(-\nu+1/2)} \tag{12}$$

$$\psi^{IV}=r^{-\nu+1/2}\mathrm{e}^{-kr}\left\{\left[\left(fr^{\nu+1/2}\mathrm{e}^{kr}\right)_{q^{-1}(\nu-1/2)}r^{-(\nu+1/2)}\mathrm{e}^{-2kr}\right]_{-1}r^{\nu-1/2}\mathrm{e}^{2kr}\right\}_{-1+q^{-1}(-\nu+1/2)} \tag{13}$$

Here, $\psi_2=\mathrm{d}^2\psi/\mathrm{d}r^2, \psi_0=\psi=\psi(r),\ f=f(r)\ (r\in\mathbb{R})$, and k, ν are given constants.

Proof. Set

$$\psi=r^{\tau}\phi \quad \phi=\phi(r) \tag{14}$$

hence, we have:

$$\phi_2 r+\phi_1 2\tau+\phi\left[\left(\tau^2-\tau+\frac{1}{4}-\nu^2\right)r^{-1}-k^2 r\right]=fr^{1-\tau} \tag{15}$$

And, we suppose τ as $\tau^2-\tau+1/4-\nu^2=0$, that is $\tau=1/2\pm\nu$.

(I) Let $\tau=\nu+1/2$. From eqs. (14) and (15) we have:

$$\psi=r^{\nu+1/2}\phi \tag{16}$$

and

$$\phi_2 r+\phi_1(2\nu+1)-\phi k^2 r = fr^{-\nu+1/2} \tag{17}$$

Next, set

$$\phi = e^{\lambda r}\varphi \quad \varphi=\varphi(r) \tag{18}$$

then, eq. (17) may be written in the form:

$$\varphi_2 r+\varphi_1(2\lambda r+2\nu+1)+\varphi[(\lambda^2-k^2)r+2\nu\lambda+\lambda] = fr^{-\nu+1/2}\mathrm{e}^{-\lambda r} \tag{19}$$

Choose λ such that $\lambda^2 - k^2 = 0$, that is $\lambda = \pm k$.
(I-i) If $\lambda = k$, we write:

$$\phi = \mathrm{e}^{kr}\varphi \tag{20}$$

and,

$$\varphi_2 r+\varphi_1(2kr+2\nu+1)+\varphi[k(2\nu+1)] = fr^{-\nu+1/2}\mathrm{e}^{-kr} \tag{21}$$

from eqs. (18) and (19). Using the ∇^α to eq. (21), we find the following equality:

$$\nabla^\alpha(\varphi_2 r)+\nabla^\alpha[\varphi_1(2kr+2\nu+1)]+\nabla^\alpha\{\varphi[k(2\nu+1)]\} = \nabla^\alpha(fr^{-\nu+1/2}\mathrm{e}^{-kr}) \tag{22}$$

Using eqs. (1)-(8) we have:

$$\nabla^\alpha(\varphi_2 r) = \varphi_{2+\alpha}r+\alpha q\varphi_{1+\alpha} \tag{23}$$

and

$$\nabla^\alpha[\varphi_1(2kr+2\nu+1)] = \varphi_{1+\alpha}(2kr+2\nu+1)+2k\alpha q\varphi_\alpha \tag{24}$$

where q is a shift operator. Making use of the relations (23) and (24), rewriting eq. (22) in the following form:

$$\varphi_{2+\alpha}r+\varphi_{1+\alpha}(2kr+2\nu+\alpha q+1)+\varphi_\alpha[k(2\alpha q+2\nu+1)] = (fr^{-\nu+1/2}\mathrm{e}^{-kr})_\alpha \tag{25}$$

Choose α such that $\alpha = -q^{-1}(\nu + 1/2)$, we have then:

$$\varphi_{2-q^{-1}(\nu+1/2)}r+\varphi_{1-q^{-1}(\nu+1/2)}(2kr+\nu+1/2) = (fr^{-\nu+1/2}\mathrm{e}^{-kr})_{-q^{-1}(\nu+1/2)} \tag{26}$$

from eq. (25).
Next, writing:

$$\varphi_{1-q^{-1}(\nu+1/2)} = u(r) \quad [\varphi = u_{-1+q^{-1}(\nu+1/2)}] \tag{27}$$

we obtain

$$u_1+u\left[2k+\frac{\nu+1/2}{r}\right] = (fr^{-\nu+1/2}\mathrm{e}^{-kr})_{-q^{-1}(\nu+1/2)}r^{-1} \tag{28}$$

from eqs. (26) and (27). A particular solution of a first-order ODE (28) is:

$$u = [(fr^{-\nu+1/2}\mathrm{e}^{-kr})_{-q^{-1}(\nu+1/2)}r^{\nu-1/2}\mathrm{e}^{2kr}]_{-1}r^{-(\nu+1/2)}\mathrm{e}^{-2kr} \tag{29}$$

Thus, we obtain the solution eq. (10) from eqs. (16), (20), (27), and (29).
(I-ii) If $\lambda = -k$, we write:

$$\phi = \mathrm{e}^{-kr}\varphi \tag{30}$$

and

$$\varphi_2 r + \varphi_1(-2kr + 2\nu + 1) + \varphi[-k(2\nu + 1)] = fr^{-\nu+1/2}\mathrm{e}^{kr} \tag{31}$$

from eqs. (18) and (19).

Using the ∇^{α} to eq. (31), we have:

$$\varphi_{2+\alpha} r + \varphi_{1+\alpha}(-2kr + 2\nu + \alpha q + 1) + \varphi_{\alpha}[-k(2\alpha q + 2\nu + 1)] = (fr^{-\nu+1/2}\mathrm{e}^{kr})_{\alpha} \tag{32}$$

Choose α such that $\alpha = -q^{-1}(\nu + 1/2)$, and replacing:

$$\varphi_{1-q^{-1}(\nu+1/2)} = \omega(r) \quad [\varphi = \omega_{-1+q^{-1}(\nu+1/2)}] \tag{33}$$

we obtain:

$$\omega_1 + \omega\left[-2k + \frac{\nu + 1/2}{r}\right] = \left(fr^{-\nu+1/2}\mathrm{e}^{kr}\right)_{-q^{-1}(\nu+1/2)} r^{-1} \tag{34}$$

from eqs. (32) and (33). A particular solution of a first-order ODE (34):

$$\omega = \left[\left(fr^{-\nu+1/2}\mathrm{e}^{kr}\right)_{-q^{-1}(\nu+1/2)} r^{\nu-1/2}\mathrm{e}^{-2kr}\right]_{-1} r^{-(\nu+1/2)}\mathrm{e}^{2kr} \tag{35}$$

Thus, we obtain the solution eq. (11) from eqs. (16), (30), (33), and (35).
(II) Let $\tau = -\nu + 1/2$. In the same way as the procedure in (I), replacing ν by $-\nu$ in (I-i) and in (I-ii), we have other solutions (12) and (13) different from (10) and (11), respectively, if $\nu \neq 0$.

Theorem 7. Let $\psi \in \{\psi : 0 \neq |\psi_{\alpha}| < \infty, \alpha \in \mathbb{R}\}$. Then the homogeneous modified Bessel equation:

$$\psi_2 + \psi\left(-k^2 + \frac{1/4 - \nu^2}{r^2}\right) = 0 \quad (r \neq 0) \tag{36}$$

has solutions of the forms:

$$\psi^{(I)} = hr^{\nu+1/2}\mathrm{e}^{kr}[r^{-(\nu+1/2)}\mathrm{e}^{-2kr}]_{-1+q^{-1}(\nu+1/2)} \tag{37}$$

$$\psi^{(II)} = hr^{\nu+1/2}\mathrm{e}^{-kr}[r^{-(\nu+1/2)}\mathrm{e}^{2kr}]_{-1+q^{-1}(\nu+1/2)} \tag{38}$$

$$\psi^{(III)} = hr^{-\nu+1/2}\mathrm{e}^{kr}[r^{\nu-1/2}\mathrm{e}^{-2kr}\%]_{-1+q^{-1}(-\nu+1/2)} \tag{39}$$

$$\psi^{(IV)} = hr^{-\nu+1/2}\mathrm{e}^{-kr}(r^{\nu-1/2}\mathrm{e}^{2kr})_{-1+q^{-1}(-\nu+1/2)} \tag{40}$$

where h is an arbitrary constant and $\nu \neq 0$.

Proof. If f = 0 under the hypotheses of *Theorem 6*, we have:

$$u_1 + u\left[2k + \frac{\nu + 1/2}{r}\right] = 0 \tag{41}$$

and

$$\omega_1 + \omega\left[-2k + \frac{\nu + 1/2}{r}\right] = 0 \tag{42}$$

for $\lambda = k$ and $\lambda = -k$, instead of eqs. (28) and (34), respectively.

Therefore, we obtain (37) for (41) and (38) for (42). And, for $\tau = -\nu + 1/2$, replacing ν by $-\nu$ in eqs. (41) and (42), we have eqs. (39) and (40).

Theorem 8. Let ψ and f are just as in *Theorem 6*. Then the non-homogeneous modified Bessel eq. (9) is satisfied by the fractional differintegrated functions:

$$\psi = \psi^{I} + \psi^{(I)}$$

Proof. It is clear by previous theorems.

Conclusion

In our study, we applied the nabla-discrete fractional calculus operator to the homogeneous and non-homogeneous modified Bessel differential equation. We obtained the discrete fractional solutions of these equations via this new operator method. For the first time, the present method was used to solve these equations in this article, and it will be applied to the similar equations in the future time.

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