AN APPLICATION OF FINITE ELEMENT METHOD FOR A MOVING BOUNDARY PROBLEM

by

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The Stefan problems called as moving boundary problems are defined by the heat equation on the domain $0 < x < s(t)$. In these problems, the position of moving boundary $s(t)$ is determined as part of the solution. As a result, they are non-linear problems and thus have limited analytical solutions. In this study, we are going to consider a Stefan problem described as solidification problem. After using variable space grid method and boundary immobilization method, collocation finite element method is applied to the model problem. The numerical solutions obtained for the position of moving boundary are compared with the exact ones and the other numerical solutions existing in the literature. The newly obtained numerical results are more accurate than the others for the time step $\Delta t = 0.0005$, it is also seen from the tables, the numerical solutions converge to exact solutions for the larger element numbers.

Key words: *variable space grid method, boundary immobilization method, collocation finite element method, cubic B-spline basis functions*

Introduction

The class of moving boundary problems has phase change which is almost concerned with mathematical models of heat, diffusion, and oxygen tension equations. The main trouble of these problems is that the position of moving boundary must be determined as part of the solution. For this reason, moving boundary problems are accepted as non-linear problems. These problems commonly called as Stefan problems have limited analytical solution for special cases. Due to the shortage of the analytical solution, some numerical methods have been developed to solve Stefan problems. In the last few years, there have been considerable interests in the numerical solutions of the Stefan problems based on front-fixing methods and front-tracking methods. Several of these applications of mentioned methods based on finite difference and finite element method (FEM) can be seen in details [1-7].

Formulation of problem

We deal with a 1-D Stefan problem which has a freezing process. In this problem, temperature distribution $U(x,t)$ and the position of moving boundary $s(t)$ are sought for whole of the domain $0 \le x \le 1$. The part of region $0 \le s_1(t) \le x \le s_2(t) \le 1$ is occupied by water and the other part of the region is occupied by ice. The initial temperature distribution in ice is symmetric about $x = 0.5$. The temperature distribution equal to –1 on fixed surfaces $x = 0$ and the ma-

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terial has phase change when the temperature of water is assumed to be zero. For initial values of moving boundaries $s_1(t) = 0.25$ and $s_2(t) = 0.75$, the position of moving boundaries must be determined along with the temperature distribution. Because of the symmetry of this problem about $x = 0.5$, it is sufficient to consider the region $0 \le x \le 0.5$ [8]. The function of temperature distribution $U(x,t)$ is governed by the heat equation:

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{1}$$

subject to the boundary conditions:

$$U(0,t) = -1, \quad U\left[s(t),t\right] = 0, \quad t > 0 \tag{2}$$

$$\frac{\partial U}{\partial x} = \frac{\mathrm{d}s(t)}{\mathrm{d}t}, \quad x = s(t), \quad t > 0 \tag{3}$$

and initial conditions:

$$s(0) = 0.25\,, \quad U(x,0) = \begin{cases} 4x - 1, & 0 \le x \le 0.25 \\ 0, & 0.25 \le x \le 0.5 \end{cases} \tag{4}$$

Recently, this problem has been solved with using a technique known as automatic differentiation. This method has been obtained by using a Taylor series expansion for the solution in which coefficients are computed by using recursive formulas derived from the heat equation itself [8].

In this paper, firstly we have used variable space grid method and boundary immobilization method to get rid of trouble the moving boundary problem. Then the numerical solutions for the position of moving boundary have been obtained by using collocation FEM based on cubic B-spline basis functions.

Cubic B-splines collocation method

Let us divide interval $[a,b]$ into N uniform element consisting of the knots such as $a = x_0 < x_1 < \ldots < x_N = b$. Cubic B-splines ϕ_m which are span $[a,b]$ are $\phi_{-1}, \phi_0, \ldots, \phi_{N+1}$. So approximate solution $U_N(x,t)$ for $U(x,t)$ is can be written:

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\phi_m(x) \tag{5}$$

where ϕ_m are trial function given by the following expressions and δ_m are time-dependent variables which will be determined from boundary and collocation conditions for the Stefan problem. The cubic B-spline is defined by the following relationship:

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3, & [x_{m-2}, x_{m-1}] \\ h^3 + 3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & [x_{m-1}, x_m] \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & [x_m, x_{m+1}] \\ (x_{m+2} - x)^3, & [x_{m+1}, x_{m+2}] \\ 0, & \text{otherwise} \end{cases}$$

where $\Delta x = h = x_m - x_{m-1}$ for all $m, m = -1, 0, 1, \ldots, N+1$. The cubic spline ϕ_m and its principle derivatives ϕ'_m and ϕ''_m disappear the outside of the interval $[x_{m-2}, x_{m+2}]$. We can tabulate values ϕ_m, $h\phi'_m$, and $h^2\phi''_m$ at the knots as given the tab. 1.

Now, we can define the finite element approximations of the nodal values for U, U' and U'' at the knot x_m [9]:

$$U = U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}$$
$$U' = U'(x_m) = -\frac{3}{h}\left(\delta_{m-1} - \delta_{m+1}\right) \tag{6}$$
$$U'' = U''(x_m) = \frac{6}{h^2}\left(\delta_{m-1} - 2\delta_m + \delta_{m+1}\right)$$

Table 1. B-spline values at the knots

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
ϕ_m	0	1	4	1	0
$h\phi'_m$	0	–3	0	3	0
$h^2\phi''_m$	0	6	–12	6	0

Variable space grid method

Murray and Landis [10] kept constant the number of space intervals between $x = 0$ and $x = s(t)$, equal to N, for all time. As a result of this, the moving boundary is always on the N^{th} grid line. Consequently, the grid size must be $\Delta x = s(t)/N$ which is changed with the time. By applying partial differentiation with respect to time t, the following expression is obtained for a given grid line $i\Delta x$. Thus, for the line $i\Delta x$, we have:

$$\left.\frac{\partial U}{\partial x}\right|_i = \left.\frac{\partial U}{\partial x}\right|_t \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_i + \left.\frac{\partial U}{\partial t}\right|_x \tag{7}$$

and also a general grid movement at the x is expressed with the equation:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{x_i}{s(t)}\frac{\mathrm{d}s(t)}{\mathrm{d}t} \tag{8}$$

By substituting eq. (8) into eq. (7), the 1-D heat equation becomes:

$$\left.\frac{\partial U}{\partial t}\right|_i = \frac{x_i}{s(t)}\frac{\mathrm{d}s(t)}{\mathrm{d}t}\left.\frac{\partial U}{\partial x}\right|_t + \left.\frac{\partial U}{\partial t}\right|_x$$

and dimensionless Stefan problem is obtained:

$$\frac{\partial U}{\partial t} = \frac{x_i}{s(t)}\frac{\mathrm{d}s(t)}{\mathrm{d}t}\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < s(t), \quad t > 0 \tag{9}$$

subject to eqs. (2)-(4) boundary and initial conditions where $s(t)$ is updated at each time step by using a suitable finite difference form of the Stefan condition $\partial U/\partial x = \mathrm{d}s(t)/\mathrm{d}t$ on $x = s(t)$. Therefore, we are going to use the following three point backward difference at the moving boundary [11]:

$$\left.\frac{\partial U}{\partial x}\right|_{x=s(t)} = \frac{3U_N - 4U_{N-1} + U_{N-2}}{2\Delta x} + O(\Delta x)^2 \tag{10}$$

and forward difference approximation for $\mathrm{d}s(t)/\mathrm{d}t$:

$$\frac{\mathrm{d}s(t)}{\mathrm{d}t} = \frac{s^{n+1} - s^n}{\Delta t}. \tag{11}$$

So, the iterative relation of moving boundary is obtained:

$$s^{n+1} = s^n + \frac{\Delta t}{2\Delta x^n}\left(3U_N^n - 4U_{N-1}^n + U_{N-2}^n\right), \quad n = 0,1,... \tag{12}$$

subject to:

$$s^0 = 0.25$$

initial condition. So, grid size $h^{n+1} = s^{n+1}/N$ is updated each time step, where N is the number of element. By substituting U and its derivatives U', U'' into eq. (9) and using Crank-Nicolson approach and time derivative of δ_m, respectively:

$$\delta_m = \frac{\delta_m^{\;n+1} + \delta_m^{\;n}}{2}, \quad \frac{\mathrm{d}\delta_m}{\mathrm{d}t} = \frac{\delta_m^{\;n+1} - \delta_m^{\;n}}{\Delta t} \tag{13}$$

we obtain a recurrence relationship between two time levels n and $n+1$ relating to two unknown parameters δ_m^{n+1} and δ_m^n:

$$a_{m1}\delta_{m-1}^{n+1} + a_{m2}\delta_m^{n+1} + a_{m3}\delta_{m+1}^{n+1} = a_{m4}\delta_{m-1}^{n} + a_{m5}\delta_m^{n} + a_{m6}\delta_{m+1}^{n}, \qquad m = 0,1,...,N \tag{14}$$

where the coefficients of the finite element system is given:

$$\begin{aligned}
\alpha_{m1} &= 1 + \frac{3\Delta t\, x_m^n (\dot{s})^n}{2s^n \Delta x^n} - \frac{3\Delta t}{(\Delta x^n)^2}, & \alpha_{m4} &= 1 - \frac{3\Delta t\, x_m^n (\dot{s})^n}{2s^n \Delta x^n} + \frac{3\Delta t}{(\Delta x^n)^2} \\
\alpha_{m2} &= 4 + \frac{6\Delta t}{(\Delta x^n)^2}, & \alpha_{m5} &= 4 - \frac{6\Delta t}{(\Delta x^n)^2} \\
\alpha_{m3} &= 1 - \frac{3\Delta t\, x_m^n (\dot{s})^n}{2s^n \Delta x^n} - \frac{3\Delta t}{(\Delta x^n)^2}, & \alpha_{m6} &= 1 + \frac{3\Delta t\, x_m^n (\dot{s})^n}{2s^n \Delta x^n} + \frac{3\Delta t}{(\Delta x^n)^2}
\end{aligned} \tag{15}$$

where s is the position of moving boundary, $\dot{s}$ – the velocity of moving boundary, Δx^n – the mesh size changing with time, and Δt – the time step. For this method, through the all numerical calculations $U_m^n \sim U(x_m^n, t_n)$ is taken as temperature distribution where $x_m^n = m\Delta x^n$, $t_n = t_0 + n\Delta t$ and t_0 is initial time.

Boundary immobilization method

This method bases on using a suitable transformation $x = \xi / s(t)$ that fixes moving boundary $x = s(t)$ as fixed boundary $\xi = 1$. Thus, physical domain $0 \le x \le s(t)$ transforms to fixed domain $0 \le \xi \le 1$. This approach firstly was used by Landau and applied to many of moving boundary problems with the aid of finite difference and FEM in recent years [1, 7, 12].

At the new co-ordinate system (ξ, t), derivatives of U and reformulated eq. (1) can be written:

$$\frac{\partial U}{\partial x} = \frac{1}{s(t)}\frac{\partial U}{\partial \xi}, \qquad \frac{\partial^2 U}{\partial x^2} = \frac{1}{s^2(t)}\frac{\partial^2 U}{\partial \xi^2}$$

$$\left.\frac{\partial U}{\partial t}\right|_x = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \left.\frac{\partial U}{\partial t}\right|_\xi = -\frac{x}{s^2(t)}\frac{\mathrm{d}s}{\mathrm{d}t} + \left.\frac{\partial U}{\partial t}\right|_\xi$$

$$\frac{\partial U}{\partial t} = \frac{\xi}{s(t)}\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\partial U}{\partial \xi} + \frac{1}{s^2(t)}\frac{\partial^2 U}{\partial \xi^2}, \qquad 0 < \xi < 1, \qquad t > 0 \tag{16}$$

subject to boundary conditions:

$$U(0,t) = -1, \quad U(1,t) = 0, \quad t > 0 \tag{17}$$

$$\frac{\partial U}{\partial \xi} = s(t)\frac{\mathrm{d}s(t)}{\mathrm{d}t}, \quad \xi = 1, \quad t > 0 \tag{18}$$

along with the initial conditions:

$$\begin{gathered} U(\xi,0) = \xi - 1, \quad 0 \le \xi \le 1, \quad t > 0 \\ s(0) = 0.25 \end{gathered} \tag{19}$$

Also for the Stefan condition (10), three point backward difference formula is written:

$$\left.\frac{\partial U}{\partial \xi}\right|_{\xi=1} = \frac{3U_N - 4U_{N-1} + U_{N-2}}{2\Delta\xi} + O(\Delta\xi)^2 \tag{20}$$

in the transformed co-ordinate system (ξ, t). So, iteration relation for the position of moving boundary is obtained:

$$s^{n+1} = s^n + \frac{\Delta t}{2\Delta\xi s(t)}\left(3U_N^n - 4U_{N-1}^n + U_{N-2}^n\right), \quad n = 0,1,... \tag{21}$$

by using eqs. (11) and (20).

For the application of FEM based on boundary immobilization method, we use the approximations of nodal values for U, U', and U'' at the knot ξ_m given with the eq. (6). By tracking similar way with variable space grid method, finite element system (14) is obtained for boundary immobilization method with the coefficients:

$$\begin{aligned} \alpha_{m1} &= 1 + \frac{3\Delta t \xi_m (\dot{s})^n}{2s^n \Delta\xi} - \frac{3\Delta t}{(s^n\Delta\xi)^2}, & \alpha_{m4} &= 1 - \frac{3\Delta t \xi_m (\dot{s})^n}{2s^n \Delta\xi} + \frac{3\Delta t}{(s^n\Delta\xi)^2} \\ \alpha_{m2} &= 4 + \frac{6\Delta t}{(s^n\Delta\xi)^2}, & \alpha_{m5} &= 4 - \frac{6\Delta t}{(s^n\Delta\xi)^2} \\ \alpha_{m3} &= 1 - \frac{3\Delta t \xi_m (\dot{s})^n}{2s^n \Delta\xi} - \frac{3\Delta t}{(s^n\Delta\xi)^2}, & \alpha_{m6} &= 1 + \frac{3\Delta t \xi_m (\dot{s})^n}{2s^n \Delta\xi} + \frac{3\Delta t}{(s^n\Delta\xi)^2} \end{aligned} \tag{22}$$

where $\Delta\xi$ is mesh size. For this method, temperature distribution $U_m^n \sim U(\xi_m, t_n)$ is calculated at the points $\xi_m = mh,\ t_n = t_0 + nk$ where t_0 is initial time and $h \equiv \Delta\xi$ is mesh size.

For the both of methods, we obtain finite element system (14) subjected to related coefficients. The mentioned system consists of $N+1$ equations but $N+3$ unknowns which are components of the vector $\delta = (\delta_{-1}, \delta_0, ..., \delta_{N+1})^T$. To solve the system uniquely we need additionally two more equations. We can obtain these equations from the boundary conditions (2)-(17):

$$\begin{aligned} \delta_{-1} &= -1 - 4\delta_0 - \delta_1 \\ \delta_{N+1} &= -4\delta_N - \delta_{N-1} \end{aligned}$$

by using these equations in the finite element system (14) and eliminating fictitious parameters δ_{-1} and δ_{N+1} for $m = 0$ and $m = N$, the system can be demonstrated in the matrix form:

$$A\delta^{n+1} = B\delta^n + r$$

where A and B are $(N+1)\times(N+1)$ tridiagonal matrixes and r is a $N+1$ column vector basing on boundary conditions. To start time evaluation of the approximate solutions δ^0 must be determined firstly. To attain δ^0 vector, we require two conditions for $U_N(x,0)$:

$$U_N(x,0)=\sum_{m=-1}^{N+1}\delta_m^0(t)\phi_m(x)$$

– the initial condition $U(x,0)$ and $U_N(x,0)$ should be equal to each other for $N+1$ points, and
– to be able to solve the system $A\delta^0=b$, we need further equations which can be obtained from the first derivatives of approximate initial conditions and both of ends points of the domain.

After we find initial vector δ^0, we get vectors $\delta^1,\delta^2,...,\delta^n$, respectively.

The stability analysis

We implement the stability analysis based on Von-Neumann theory in which the growth factor of typical Fourier mode is defined:

$$\delta_m^n(t)=\zeta^n e^{im\beta h}$$

where β is the mode number and h is element size determined from the numerical scheme. By substituting this equation in the finite element system (14) and performing some simplification operations we have:

$$\zeta=\frac{a_1+ib}{a_2-ib}$$

where

$$a_1=\left(2+\frac{6\Delta t}{h^2}\right)\cos\beta h+\left(4-\frac{6\Delta t}{h^2}\right)$$

$$a_2=\left(2-\frac{6\Delta t}{h^2}\right)\cos\beta h+\left(4+\frac{6\Delta t}{h^2}\right)$$

$$b=\left[\frac{3\Delta t(\dot{s})^n x_m^n}{hs^n}\right]\sin\beta h$$

for FEM based on variable space grid method (VSG-FEM) and:

$$a_1=\left[2+\frac{6\Delta t}{(s^n h)^2}\right]\cos\beta h+\left[4-\frac{6\Delta t}{(s^n h)^2}\right]$$

$$a_2=\left[2-\frac{6\Delta t}{(s^n h)^2}\right]\cos\beta h+\left[4+\frac{6\Delta t}{(s^n h)^2}\right]$$

$$b=\left[\frac{3\Delta t(\dot{s})^n \xi_m}{hs^n}\right]\sin\beta h$$

for FEM based on boundary immobilization method (BIM-FEM). For the stability of these methods, the growth factor must satisfy $|\zeta|\le 1$. If we do essential processing we will see that the system of (14) is unconditionally stable for both of the methods. It should be pointed that h shows Δx^n for VSG-FEM and ξ for BIM-FEM.

Numerical results and conclusion

The problem described with the eqs. (1)-(4) has been solved by Asaithambi [8] and Finn and Varoglu [13]. Finn and Varoglu [13] have used FEM. Then, Asaithambi [8] has used automatic differentiation method formed with fourth order Taylor series and obtained better numerical results than that of Finn and Varoglu [13]. In this paper, present numerical results will be compared with these numerical solutions and exact solutions of Rubinstein [14] for the position of the moving boundary.

In tab. 2, numerical results obtained by using VSG-FEM and BIM-FEM are compared with the other numerical solutions [8, 13] and exact solutions [14] for the position of moving boundary at the element number $N = 10$ and time step $\Delta t = 0.0005$. It is clearly seen that the numerical results obtained by the present methods are in excellent agreement with the exact solution and more accurate than the other numerical solutions.

Table 2. Comparison of numerical results for the position of moving boundary at the parameters $N = 10$ and $\Delta t = 0.0005$

	VSG-FEM	BIM-FEM	Rubinstein [14]	Asaithambi [8]	Finn and Varoglu [13]
0.01	0.281637	0.281666	0.281347	0.2818	0.2806
0.02	0.308179	0.308231	0.307925	0.3083	0.3072
0.03	0.332342	0.332411	0.332077	0.3325	0.3315
0.04	0.354798	0.354880	0.354519	0.3549	0.3541
0.06	0.395863	0.395961	0.395495	0.3960	0.3956
0.08	0.433036	0.433144	0.432581	0.4331	0.4333
0.10	0.467255	0.467370	0.466754	0.4674	0.4682

In tab. 3 the numerical results obtained using present method are compared with other numerical solutions and exact solutions for the position of moving boundary at the element number $N = 20$ and time step $\Delta t = 0.0001$. It can be seen that the present results are in good agreement the other numerical solutions and exact ones.

Table 3. Comparison of numerical results for the position of moving boundary at the parameters $N = 20$ and $\Delta t = 0.0001$

	VSG-FEM	BIM-FEM	Rubinstein [14]	Asaithambi [8]	Finn and Varoglu [13]
0.01	0.281369	0.281375	0.281347	0.2814	0.2806
0.02	0.307877	0.307887	0.307925	0.3079	0.3072
0.03	0.332022	0.332036	0.332077	0.3320	0.3315
0.04	0.354465	0.354481	0.354519	0.3545	0.3541
0.06	0.395509	0.395529	0.395495	0.3955	0.3956
0.08	0.432667	0.432687	0.432581	0.4327	0.4333
0.10	0.466871	0.466894	0.466754	0.4669	0.4682

In tabs. 4 and 5, to show efficiency of FEM based on VSG-FEM and BIM-FEM, the position of moving boundary are given for the larger element numbers and fixed time step $\Delta t = 0.0005$. It is clearly seen that for the smaller mesh sizes the position of the moving boundary get closer to exact values of the moving boundary.

In this paper, firstly VSG and BIM were used to get rid of the trouble of the model moving boundary problem described as solidification problem. Then, FEM constructed with cubic B-splines was applied to solve the model problem numerically. The computational results compared with the other numerical solutions and exact solutions for the position of moving boundary. We have been reasonably more accurate solutions than the other numerical solutions for the parameters $N = 10$ and $\Delta t = 0.0005$. Furthermore, FEM attains to expected convergence for the larger element numbers.

Table 4. The position of moving boundary for VSG-FEM at different element numbers and Δt = 0.0005

t	N = 10	N = 20	N = 40	N = 80	Rubinstein [14]
0.01	0.281637	0.281547	0.281522	0.281513	0.281347
0.02	0.308179	0.308073	0.308046	0.308038	0.307925
0.03	0.332342	0.332225	0.332196	0.332188	0.332077
0.04	0.354798	0.354671	0.354641	0.354633	0.354519
0.06	0.395863	0.395717	0.395683	0.395674	0.395495
0.08	0.433036	0.432872	0.432834	0.432825	0.432581
0.10	0.467255	0.467076	0.467035	0.467026	0.466754

Table 5. The position of moving boundary for BIM-FEM at different element numbers and Δt = 0.0005

t	N = 10	N = 20	N = 40	N = 80	Rubinstein [14]
0.01	0.281666	0.281576	0.281551	0.281542	0.281347
0.02	0.308231	0.308125	0.308098	0.308090	0.307925
0.03	0.332411	0.332294	0.332265	0.332257	0.332077
0.04	0.354880	0.354752	0.354722	0.354714	0.354519
0.06	0.395961	0.395814	0.395780	0.395771	0.395495
0.08	0.433144	0.432980	0.432942	0.432933	0.432581
0.10	0.467370	0.467191	0.467191	0.467149	0.466754

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