APPLICATIONS OF A NOVEL INTEGRAL TRANSFORM TO THE CONVECTION-DISPERSION EQUATIONS

by

Xin LIANG[a], Guannan LIU[b*], and Shanjie SU[a]

[a] State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou, China
[b] School of Mechanics and Civil Engineering, China University of Mining and Technology, Xuzhou, China

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In this paper, we extend the novel integral transform of some functions by the duality relationship between it and Laplace transform. Additionally, applying the novel integral transform, we solve a 1-D convection-dispersion equation describing the dispersion process of chemical additives in porous rocks during the hydraulic fracturing. The results indicate that the novel integral transform can provide a new idea to obtain more exact solutions of different convection-dispersion problems.

Key words: *integral transform, convection-dispersion equation, hydraulic fracturing*

Introduction

The convection-dispersion equations (CDE) are widely applied to describe the transport of suspended particles in nature porous media, which is touched upon many engineering branches of the environment, the architecture, as well as the energy [1-3]. In particular, for energy exploitation (such as shale gas exploitation), with the massive application of hydraulic fracturing in recent years [4-6], the problem of the groundwater pollution has brought about the widespread attentions due to the fact that the hydraulic fracturing fluid containing numerous chemical additives is dispersed into the aquifer. Therefore, it is of a critical importance for understanding the dispersion process of the chemicals in aquifer [7, 8]. Usually, the analytic solutions of the CDE can be intuitively used to provide quantitative analysis of the dispersion of particles.

Presently, many researchers have solved the CDE with special initial and boundary conditions by the integral transforms (IT), such as Laplace transform (LT), and Fourier transform (FT) [9]. However, with the development of the IT, some novel IT can be applied to provide new ideas for obtaining the analytic solutions of the CDE. Recently, some novel IT were proposed as the extensions of classical FT and LT to solve the heat transfer equations and diffusion equations [10-15]. Particularly, in [12], the duality relationship between a novel integral transform (NIT) and LT was studied in detail and the NIT was proved to be effective to solve some PDE like LT. This indicates that the NIT can be applied to obtain the exact solutions of the CDE.

* Corresponding author, e-mail: guannanliu@cumt.edu.cn

In view of the analysis, we plan to extend the NIT of some functions by the duality characteristic between the NIT and LT. Meanwhile, the 1-D CDE is analyzed by the NIT and the corresponding solution is discussed graphically.

The NIT of some basic functions

According to the idea [10], the definition of the NIT is described:

$$\Omega(\zeta) = N_I\left[\phi(x)\right] = \int_0^\infty \phi(x)\frac{e^{-\zeta x}}{\zeta}dx, \quad \zeta > 0 \tag{1}$$

where $\phi(x)$, $x > 0$, is a real function, $e^{-\zeta x}/\zeta$ is the kernel function, and N_I is the operator of NIT.

$$F(\zeta) = L\left[\phi(x)\right] = \int_0^\infty \phi(x)e^{-\zeta x}dx, \quad \zeta > 0 \tag{2}$$

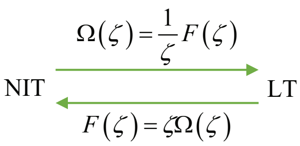

Figure 1. The duality of the NIT and LT

where $\phi(x)$, $x > 0$, is a real function, $e^{-\zeta x}$ is the kernel function, and L is the operator of the LT.

Obviously, from previous definitions, the NIT is similar to the LT in form. The main difference between them is reflected in the selection of the kernel function [16]. It is indicated that the NIT and LT are correlated, as well as mutually different. In order to verify the applicability of duality relationship as shown in fig. 1, we derive the NIT of five basic functions.

In detail, the NIT of five functions are proved as follows.

– For $\phi(x) = 1$ and $x > 0$, we have [17]:

$$F(\zeta) = L[1] = \frac{1}{\zeta} \tag{3}$$

According to the duality relationship in fig. 1, we have the following equation:

$$\Omega(\zeta) = \frac{1}{\zeta}F(\zeta) = \frac{1}{\zeta}L[1] = \frac{1}{\zeta^2} \tag{4}$$

– For $\phi(x) = x$ and $x > 0$, we have [18]:

$$F(\zeta) = L[x] = \frac{1}{\zeta^2} \tag{5}$$

The NIT of $\phi(x)$ can be written:

$$\Omega(\zeta) = \frac{1}{\zeta}F(\zeta) = \frac{1}{\zeta}L[x] = \frac{1}{\zeta^3} \tag{6}$$

– For $\phi(x) = e^{ax}$ and $x > 0$, we have [17]:

$$F(\zeta) = L\left[e^{ax}\right] = \frac{1}{\zeta - a} \tag{7}$$

The NIT of $\phi(x)$ is derived by the duality relationship:

$$\Omega(\zeta)=\frac{1}{\zeta}F(\zeta)=\frac{1}{\zeta}L\left[\mathrm{e}^{ax}\right]=\frac{1}{\zeta(\zeta-a)} \tag{8}$$

– For $\phi(x)=\sin(ax)$ and $x>0$, we have [17]:

$$F(\zeta)=L\left[\sin(ax)\right]=\frac{a}{\zeta^2+a^2} \tag{9}$$

The NIT of $\phi(x)$ becomes:

$$\Omega(\zeta)=\frac{1}{\zeta}F(\zeta)=\frac{1}{\zeta}L\left[\sin(ax)\right]=\frac{a}{\zeta\left(\zeta^2+a^2\right)} \tag{10}$$

– For $\phi(x)=\cos(ax)$ and $x>0$, we have [17]:

$$\Omega(\zeta)=\frac{1}{\zeta}F(\zeta)=\frac{1}{\zeta}L\left[\cos(ax)\right]=\frac{\zeta}{\left(\zeta^2+a^2\right)} \tag{11}$$

The NIT of $\phi(x)$ is:

$$\Omega(\zeta)=\frac{1}{\zeta}F(\zeta)=\frac{1}{\zeta}L\left[\cos(ax)\right]=\frac{1}{\left(\zeta^2+a^2\right)} \tag{12}$$

The results of eqs. (4), (6), (8), (10), and (12) are same as those given in [10]. It is illustrated that the duality relationship between the NIT and LT is applicable to obtain the NIT of more functions. Thus, we extend the NIT of more functions in the appendix and adopt them to solve the CDE in next section.

Solving the CDE by the NIT

As displayed in fig. 2, the aquifer is assumed to be a semi-infinite porous medium and contains no chemical additives existing in fracturing fluid in initial state. During hydraulic fracturing, the fracturing fluid flows into the aquifer with the constant velocity, u. Meanwhile, the concentration of chemical additives in aquifer boundary is treated as a constant, C_0. Thus, the dispersion process of chemical additives in the aquifer can be described:

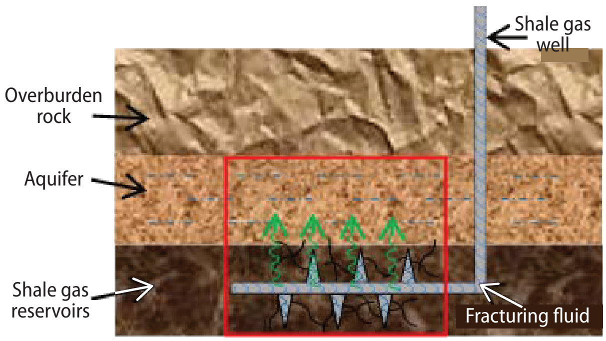

Figure 2. The schematic diagram of hydraulic fracturing

$$k\frac{\partial^2 c(x,t)}{\partial x^2}-u\frac{\partial c(x,t)}{\partial x}=\frac{\partial c(x,t)}{\partial t} \tag{13}$$

where $c(x,t)$ is the concentration of the chemical additives in the aquifer, k – the dispersion coefficient, and u – the average travel velocity of the fracturing fluid.

The initial-value condition (IVC) and the boundary-value conditions (BVC) are:

$$\begin{cases} c(x,0)=0 \ \text{(IVC)} \\ c(0,t)=C_0, \ c(\infty,t)=0 \ \text{(BVC)} \end{cases}$$

respectively, where C_0 is a real constant.

According to the properties of the NIT in [10], the NIT of eq. (13) with respect to *t* is obtained as:

$$k\frac{\partial^2 C(x,\zeta)}{\partial x^2}-u\frac{\partial C(x,\zeta)}{\partial x}=\zeta C(x,\zeta)-\frac{c(x,0)}{\zeta} \tag{14}$$

Correspondingly, the NIT of BVC becomes:

$$C(0,\zeta)=\frac{C_0}{\zeta^2}, \ C(\infty,\zeta)=0 \tag{15}$$

Substitution of the IVC into eq. (14) results in:

$$k\frac{\partial^2 C(x,\zeta)}{\partial x^2}-u\frac{\partial C(x,\zeta)}{\partial x}-\zeta C(x,\zeta)=0 \tag{16}$$

From the eq. (16), we can obtain its eigenvalue equation [18]:

$$k\lambda^2-u\lambda-\zeta=0 \tag{17}$$

Then, the eigenvalues of eq. (17) are given:

$$\lambda_1=\frac{u+\sqrt{u^2+4k\zeta}}{2k}, \ \lambda_2=\frac{u-\sqrt{u^2+4k\zeta}}{2k} \tag{18}$$

Thus, the solution of eq. (16) is presented:

$$C(x,\zeta)=m_1\mathrm{e}^{\frac{u+\sqrt{u^2+4k\zeta}}{2k}x}+m_2\mathrm{e}^{\frac{u-\sqrt{u^2+4k\zeta}}{2k}x} \tag{19}$$

Substituting the BVC eq. (15) into eq. (19), we have:

$$m_1+m_2=\frac{C_0}{\zeta^2}, \quad m_1=0 \tag{20}$$

Taking eq. (20) into the eq. (19), we obtain:

$$C=\frac{C_0}{\zeta^2}\mathrm{e}^{\left(\frac{u-\sqrt{u^2+4k\zeta}}{2k}\right)x}=\frac{C_0}{\zeta^2}\mathrm{e}^{\frac{ux}{2k}}\mathrm{e}^{-\frac{\sqrt{u^2+4k\zeta}}{2k}x} \tag{21}$$

According to the result in tab. 1 (*Appendix*), we have the NIT of a special function:

$$N_I\left[\frac{1}{2}\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{a}{t}}-\sqrt{bt}\right)(\mathrm{e}^{-\sqrt{ab}}+\mathrm{e}^{\sqrt{ab}})\right]=\frac{1}{\zeta^2}\mathrm{e}^{-\sqrt{a\zeta+ab}} \tag{22}$$

where *a*, *b* are the constants, and $\mathrm{erfc}[1/2(a/t)^{1/2}-(bt)^{1/2}]$ is the error function [17].

Finally, we solve the inverse NIT of eq. (21) with respect to t and have:

$$C=\frac{C_0}{2}\left\{\mathrm{e}^{\frac{ux}{k}}\mathrm{erfc}\left(\frac{x+ut}{2\sqrt{kt}}\right)+\mathrm{erfc}\left(\frac{x-ut}{2\sqrt{kt}}\right)\right\} \tag{23}$$

The convection-dispersion solutions of eq. (23) for varied parameters are illustrated in figs. 3(a) and 3(b). The results show that the dispersion ranges of chemical additives will be wider and wider as the time increases and the concentrations of chemical additives in aquifer also will increase. For example, fig. 3(a) demonstrates that the dispersion ranges reach approximately 25 meters in 60 seconds. In addition, 70% chemical additives are discovered at 30 meters in 60 minutes [red curve (1) in fig. 3(b)], while approximately 0 in 60 seconds [red curve (1) in fig. 3(a)].

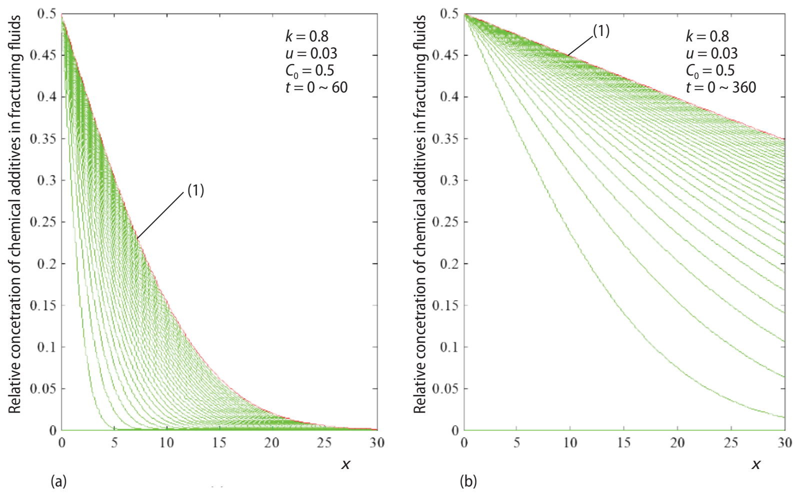

Figure 3. The convection-dispersion curves with different parameters: (a) 2 seconds time interval for each curve, (b) 2 minutes time interval for each curve; red curve (1)
(*for color image see journal web site*)

Conclusion

In this study, the NIT was applied to obtain the exact solution of a 1-D CDE for an efficient description of the dispersion process of chemical additives in porous rocks during the hydraulic fracturing. With the help of the duality relationship between the NIT and LT, the extended NIT table presented in the *Appendix* was convenient to obtain the solutions of some PDE. The results were given to show that the dispersion range of chemical additives in the aquifer is wide (25 meters in 60 seconds) and that the dispersion scale is large (70% within 60 minutes). It is demonstrated that the chemical additives affect the groundwater quality with the time increases due to the strong dispersion process.

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Nomenclature

c – concentration of the chemical additives, [gL^{-1}]
k – dispersion coefficient, [cm^2s^{-1}]
t – time, [s]
u – flow velocity, [cms^{-1}]
x – diffusion distances, [m]

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Appendix

Table 1. The NIT of some functions

	$\phi(x)$	$F(\zeta)=L[\phi(x)]$
1	$\sinh(ax)$	$\dfrac{a}{\zeta(\zeta^2-a^2)}$
2	$\cosh(ax)$	$\dfrac{1}{\zeta^2-a^2}$
3	$x^n e^{-ax}$	$\dfrac{\Gamma(n+1)}{\zeta(\zeta+a)^{(n+1)}}$
4	$\sin(ax)e^{bx}$	$\dfrac{a}{\zeta\left[(\zeta-b)^2+a^2\right]}$
5	$\cos(ax)e^{bx}$	$\dfrac{\zeta-b}{\zeta\left[(\zeta-b)^2+a^2\right]}$
6	$x\sin(ax)$	$\dfrac{2a}{(\zeta^2+a^2)^2}$
7	$x\cos(ax)$	$\dfrac{\zeta^2-a^2}{\zeta(\zeta^2+a^2)^2}$
8	$x\sinh(ax)$	$\dfrac{2a}{(\zeta^2-a^2)^2}$
9	$x\cosh(ax)$	$\dfrac{\zeta^2+a^2}{\zeta(\zeta^2-a^2)^2}$
10	$\dfrac{1}{\sqrt{\pi x}}(1+2ax)e^{ax}$	$\dfrac{1}{(\zeta-a)\sqrt{\zeta-a}}$
11	$\dfrac{1}{\sqrt{\pi x}}\sin\left(2\sqrt{ax}\right)$	$\dfrac{1}{\zeta^{5/2}}e^{-a/\zeta}$
12	$\dfrac{1}{\sqrt{\pi x}}\cos\left(2\sqrt{ax}\right)$	$\dfrac{1}{\zeta^{3/2}}e^{-a/\zeta}$
13	$\dfrac{1}{\sqrt{\pi a}}\sinh\left(2\sqrt{ax}\right)$	$\dfrac{1}{\zeta^{5/2}}e^{a/\zeta}$
14	$\dfrac{1}{\sqrt{\pi a}}\cosh\left(2\sqrt{ax}\right)$	$\dfrac{1}{\zeta^{3/2}}e^{a/\zeta}$
15	$\mathrm{erf}\left(\dfrac{a}{2\sqrt{x}}\right)$	$\dfrac{1}{\zeta^2}\left(1-e^{-a\sqrt{\zeta}}\right)$

→

Table 1. (continuation)

	$\phi(x)$	$F(\zeta)=L[\phi(x)]$
16	$e^{a^2x}\operatorname{erf}\left(a\sqrt{x}\right)$	$\dfrac{a}{\zeta^{3/2}\left(\zeta-a^2\right)}$
17	$\operatorname{erfc}\left(\dfrac{a}{2\sqrt{x}}\right)$	$\dfrac{1}{\zeta^2}e^{-a\sqrt{\zeta}}$
18	$\dfrac{1}{\sqrt{\pi x}}+ae^{a^2x}\operatorname{erf}\left(a\sqrt{x}\right)$	$\dfrac{1}{\sqrt{\zeta}\left(\zeta-a^2\right)}$
19	$\dfrac{1}{2}\left[e^{-\sqrt{ab}}\operatorname{erfc}\left(\dfrac{\sqrt{a}-2\sqrt{b}x}{2\sqrt{x}}\right)+e^{\sqrt{ab}}\operatorname{erfc}\left(\dfrac{\sqrt{a}+2\sqrt{b}x}{2\sqrt{x}}\right)\right]$	$\dfrac{1}{\zeta^2}e^{-\sqrt{a(\zeta+b)}}$
20	$\dfrac{1}{2}\left[e^{-ab}\operatorname{erfc}\left(\dfrac{b-2ax}{2\sqrt{x}}\right)+e^{ab}\operatorname{erfc}\left(\dfrac{b+2ax}{2\sqrt{x}}\right)\right]$	$\dfrac{1}{\zeta}e^{-b\sqrt{\zeta+a^2}}$

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