

A NON-DIFFERENTIABLE SOLUTION FOR THE LOCAL FRACTIONAL TELEGRAPH EQUATION

by

**Jie LI^a, Ce ZHANG^{b*}, Weixing LIU^a, Yuzhu ZHANG^a,
Aimin YANG^a, and Ling ZHANG^c**

^a The Ministry of Education Key Laboratory with Modern Metallurgical Technology, North China University of Science and Technology, Tangshan, China

^b School of Computer Science and Technology, Harbin Institute of Technology at Weihai, Weihai, China

^c Development Planning Department, Yanching Institute of Technology, Sanhe, China

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In this paper, we consider the linear telegraph equations with local fractional derivative. The local fractional Laplace series expansion method is used to handle the local fractional telegraph equation. The analytical solution with the non-differentiable graphs is discussed in detail. The proposed method is efficient and accurate.

Key words: *telegraph equation, analytical solution, local fractional derivative, water wave, local fractional Laplace series expansion method*

Introduction

Local fractional calculus [1] has played an important role in the field of mathematical science and mathematical physics, such as the generalized convex [2] and s-convex [3, 4] functions on fractal sets, and Pompeiu-type [5], Steffensen [6], Hermite-Hadamard [7], Holder [8], Hilbert [9], Korteweg-de Vries [10], Burgers [11], Boussinesq [12], heat conduction [13], diffusion [14, 15], tricomis [16], Goursat [17], and others [18-21].

The many non-differentiable problems were described by the local fractional ODE and PDE. Many computational techniques, such as the non-differentiable travelling-wave transformation technology [10-12], local fractional variational iteration Laplace transform method [13], decomposition method [14], series expansion sumudu transform method [15], local fractional integral transform method [19], and Laplace series expansion method [22-25], were developed to find the solutions for the local fractional ODE and PDE.

The local fractional linear telegraph equations were solved by the Laplace decomposition method [26] and Laplace variational iteration method [27]. The local fractional Laplace series expansion method has not yet been considered to handle the local fractional linear telegraph equation. In the sprint of the idea, the main target of the paper is to present a new application of the local fractional Laplace series expansion method to find the analytic solution of the local fractional linear telegraph equation.

* Corresponding author, e-mail: zhangce@hitwh.edu.cn

Mathematical fundamentals

In this section, we introduce the concepts of the local fractional derivative and integral and present the local fractional Laplace type transform method.

The local fractional derivative operator of $\Omega(\tau)$ is defined [1-27]:

$$\Omega^{(\alpha)}(t_0) = \left. \frac{d^\alpha \Omega(t)}{dt^\alpha} \right|_{t=t_0} = \lim_{t \rightarrow t_0} \frac{\Delta^\alpha [\Omega(t) - \Omega(t_0)]}{(t - t_0)^\alpha} \tag{1}$$

where $\Delta^\alpha [\Omega(t) - \Omega(t_0)] \cong \Gamma(1 + \alpha) \Delta [\Omega(t) - \Omega(t_0)]$.

The properties of the local fractional derivative operator [1] are listed in tab. 1.

As the inverse operator of eq. (1), the local fractional integral operator of $\theta(\tau)$ is defined [1, 10, 25]:

$$\Omega(t) = {}_a I_b^{(\alpha)} \theta(t) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b \theta(t) (dt)^\alpha = \frac{1}{\Gamma(1 + \alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} \theta(t) (\Delta t)^\alpha \tag{2}$$

where $\Delta t = t_{j+1} - t_j$, $j = 0, \dots, N - 1$, $t_0 = a$, and $t_N = b$.

Table 1. The local fractional derivatives of the special functions defined on Cantor sets

Special functions	Local fractional derivatives
$\frac{t^{k\alpha}}{\Gamma(1 + k\alpha)} \quad (k \in \mathbb{N})$	$\frac{t^{(k-1)\alpha}}{\Gamma[1 + (k-1)\alpha]}$
$E_\alpha(t^\alpha)$	$E_\alpha(t^\alpha)$
$\sin_\alpha(t^\alpha)$	$\cos_\alpha(t^\alpha)$
$\cos_\alpha(t^\alpha)$	$-\sin_\alpha(t^\alpha)$

Table 2. The local fractional integrals of the special functions

Special functions $\theta(t)$	Local fractional integrals ${}_0 I_t^{(\alpha)} \theta(t)$
$\frac{t^{(k-1)\alpha}}{\Gamma[1 + (k-1)\alpha]} \quad (k \in \mathbb{N})$	$\frac{t^{k\alpha}}{\Gamma(1 + k\alpha)}$
$E_\alpha(t^\alpha)$	$E_\alpha(t^\alpha) - 1$
$\cos_\alpha(t^\alpha)$	$\sin_\alpha(t^\alpha)$
$-\sin_\alpha(t^\alpha)$	$\cos_\alpha(t^\alpha) - 1$

The properties of the local fractional integral operator [1] are listed in tab. 2.

The local fractional Laplace type transform operator of $\Omega(\tau)$ is defined [1]:

$$\tilde{Y}_\alpha \{ \Omega(t) \} = \Omega_{y^{\tilde{y}, \alpha}}^{\tilde{y}, \alpha}(y) = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha(-y^\alpha t^\alpha) \Omega(t) (dt)^\alpha, \quad 0 < \alpha \leq 1 \tag{3}$$

The inverse local fractional Laplace type transform operator is defined [1]:

$$\Omega(t) = \tilde{Y}_\alpha^{-1} \{ \Omega_{y^{\tilde{y}, \alpha}}^{\tilde{y}, \alpha}(y) \} = \frac{1}{(2\pi)^\alpha} \int_{\mu - i\infty}^{\mu + i\infty} E_\alpha(y^\alpha \tau^\alpha) \Omega_{y^{\tilde{y}, \alpha}}^{\tilde{y}, \alpha}(y) (dy)^\alpha \tag{4}$$

where $y^\alpha = \mu^\alpha + i^\alpha \infty^\alpha$ and $\text{Re}(y^\alpha) = \mu^\alpha$.

Some properties of the local fractional Laplace type transform operator [1] are:

$$\tilde{Y}_\alpha \{ \Omega_1(t) + \Omega_2(t) \} = \tilde{Y}_\alpha \{ \Omega_1(t) \} + \tilde{Y}_\alpha \{ \Omega_2(t) \} \tag{5}$$

$$\tilde{Y}_\alpha \{ \Omega^{(n\alpha)}(t) \} = y^{n\alpha} \tilde{Y}_\alpha \{ \Omega(t) \} - \sum_{k=1}^n y^{(k-1)\alpha} \Omega^{(n-k)\alpha}(0) \tag{6}$$

$$\tilde{Y}_\alpha \{E_\alpha(t^\alpha)\} = \frac{1}{y^\alpha - 1} \quad (7)$$

$$\tilde{Y}_\alpha \left\{ \frac{\tau^{k\alpha}}{\Gamma(1+k\alpha)} \right\} = \frac{1}{y^{(k+1)\alpha}} \quad (8)$$

When c is a constant, the properties of the local fractional Laplace type transform operator [1] are listed in tab. 3.

The computational technique applied

In this section, we introduce the basic ideas of the local fractional series expansion and Laplace series expansion methods.

The series expansion method

Following the idea in [8], we consider the local fractional PDE in the operator form:

$$\phi_t^{(n\alpha)} = \Pi_\alpha \phi \quad (9)$$

where $\phi_t^{(n\alpha)} = \partial^{n\alpha} \phi(x, g)/dt^{n\alpha}$ and Π_α is a linear local operator with respect to x .

Let us consider:

$$\phi(x, t) = \sum_{i=0}^{\infty} \Lambda_i(t) \omega_i(x) \quad (10)$$

where $\Lambda_i(t)$ and $\omega_i(x)$ are the non-differentiable functions.

Suppose that $\Lambda_i(t) = t^{i\alpha} / \Gamma(1+i\alpha)$, then we have:

$$\phi(x, t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(x) \quad (11)$$

such that

$$\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_{i+n}(x) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (\Pi_\alpha \omega_i)(x) \quad (12)$$

From eq. (12) we have:

$$\omega_{i+n}(x) = (\Pi_\alpha \omega_i)(x) \quad (13)$$

with the initial condition:

$$\omega_0(x) = \phi(x, 0) \quad (14)$$

Finally, after determining the terms of $\omega_i(x)$, we obtain the solution of eq. (10) in the series form:

$$\phi(x, \tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(x) \quad (15)$$

Table 3. The local fractional Laplace type transforms of the special functions

Special functions $\Omega(t)$	Local fractional Laplace type transforms $\tilde{Y}_\alpha \{\Omega(t)\} = \Omega_y^{\tilde{Y}, \alpha}(y)$
$\frac{\tau^{k\alpha}}{\Gamma(1+k\alpha)}$	$\frac{1}{y^{(k+1)\alpha}}$
$E_\alpha(ct^\alpha)$	$\frac{1}{y^\alpha - c}$
$\cos_\alpha(ct^\alpha)$	$\frac{y^\alpha}{y^{2\alpha} + c^2}$
$\sin_\alpha(ct^\alpha)$	$\frac{c}{y^{2\alpha} + c^2}$

The local fractional Laplace series expansion method

With the aid of the idea of the local fractional Laplace series expansion method [8], considering:

$$\Lambda_i(t) = \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \quad (16)$$

and

$$\phi(x,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(x) \quad (17)$$

we can write eq. (9):

$$\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_{i+n}(x) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (\Pi_{\alpha} \omega_i)(x) \quad (18)$$

which leads to:

$$\omega_{i+n}(y) = (\Pi_{\alpha} \omega_i)(y) \quad (19)$$

where

$$\tilde{Y}_{\alpha} \{ \phi(x,t) \} = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(y) \quad (20)$$

$$\tilde{Y}_{\alpha} \{ \phi_t^{(n\alpha)}(x,t) \} = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_{i+n}(y) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_{i+n}(y) \quad (21)$$

$$\tilde{Y}_{\alpha} \{ \Pi_{\alpha} \phi(x,t) \} = \Pi_{\alpha} \left[\sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(y) \right] = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} (\Pi_{\alpha} \omega_i)(y) \quad (22)$$

Thus, we have the following iteration equation:

$$\begin{cases} \omega_{i+2}(y) = (\Pi_{\alpha} \omega_i)(y) \\ \omega_0(y) = \tilde{Y}_{\alpha} \{ \phi_0(x,0) \} = \phi_0(y,0) \\ \omega_1(y) = \tilde{Y}_{\alpha} \{ \phi_1(x,0) \} = \phi_1(y,0) \end{cases} \quad (23)$$

and so on, where $\phi_i(x,0)$ are the initial value conditions.

Thus, we have:

$$\phi(y,t) = \sum_{i=0}^{\infty} \frac{t^{i\alpha}}{\Gamma(1+i\alpha)} \omega_i(y) \quad (24)$$

which deduces that:

$$\phi(x,t) = \tilde{Y}_{\alpha}^{-1} \{ \phi(y,t) \} = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)} \tilde{Y}_{\alpha}^{-1} \{ \omega_i(y) \} \quad (25)$$

Solving the local fractional telegraph equation

Let us consider the local fractional telegraph equation [26]:

$$\frac{\partial^{2\alpha}\phi(x,t)}{\partial t^{2\alpha}} = \frac{\partial^{2\alpha}\phi(x,t)}{\partial x^{2\alpha}} + \frac{\partial^\alpha\phi(x,t)}{\partial x^\alpha} - \phi(x,t) \quad (26)$$

subject to the initial conditions:

$$\phi(0,x) = E_\alpha(-x^\alpha) \quad (27)$$

$$\frac{\partial^\alpha\phi(0,x)}{\partial t^\alpha} = E_\alpha(-x^\alpha) \quad (28)$$

With the help of the idea of the local fractional Laplace series expansion method, we structure:

$$\omega_{i+2}(x) = (\Pi_\alpha \omega_i)(x) \quad (29)$$

where

$$(\Pi_\alpha \omega_i) = \frac{d^{2\alpha}\omega(x)}{dx^{2\alpha}} + \frac{\partial^\alpha\omega(x)}{dx^\alpha} + \beta\omega(x) \quad (30)$$

which leads to:

$$\omega_{i+2}(y) = (\Lambda^\alpha \omega_i)(y) \quad (31)$$

where

$$(\Lambda^\alpha \omega_i)(y) = \tilde{Y}_\alpha \left\{ \frac{d^{2\alpha}\omega_i(x)}{dx^{2\alpha}} + \frac{\partial^\alpha\omega_i(x)}{dx^\alpha} + \beta\omega_i(x) \right\} \quad (32)$$

$$\omega_0(y) = \frac{1}{y^\alpha + 1} \quad (33)$$

$$\omega_1(y) = \frac{1}{y^\alpha + 1} \quad (34)$$

Thus, finally, we have:

$$\phi(x,t) = E_\alpha(-x^\alpha) \left[\sin_\alpha(t^\alpha) + \cos_\alpha(t^\alpha) \right] \quad (35)$$

which is in agreement with the result obtained in [26].

Conclusion

In the present work, the local fractional Laplace series expansion method was implemented to present the non-differentiable solution for the local fractional telegraph equation. The method is accurate and efficient.

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Nomenclature

x – space co-ordinate, [m]
 t – time, [s]

Greek symbols

α – fractal dimensional order, [-]
 $\phi(x,t)$ – concentration, [ms^{-1}]

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