

## APPROXIMATE SOLUTION FOR BURGERS EQUATION WITH LOCAL FRACTIONAL DERIVATIVE BY YANG-LAPLACE DECOMPOSITION METHOD

by

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*We presented the application of local fractional Yang-Laplace decomposition method to a local fractional Burgers equation. Our results show that the method gives high accuracy series solutions that converge very rapidly.*

*Key words: local fractional Burgers equations, local fractional derivative, Yang-Laplace decomposition method*

### Introduction

With the development of the fractional calculus theory, it has been found that many non-differentiable phenomena in real world can be described by using non-linear local fractional differential equations [1-7]. In most cases, the local fractional differential equations were applied to model problems in fractal mathematics and engineering. They have attracted lots of attention among scientists [8-15]. Finding non-differentiable solutions is the hot topics. But, in general, it is difficult to obtain an exact analytic solution for a non-linear local fractional differential equation. Some approximate methods have largely been used to handle these equations [16-20].

Recently, some useful techniques have been successfully applied to local fractional differential equations. The main techniques include the decomposition method and Yang-Laplace transform method with local fractional operator, see [1, 16-20].

In this paper, our aim is to use the local fractional Yang-Laplace decomposition method to solve the following non-linear local fractional Burgers equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au \frac{\partial u}{\partial x} - b \frac{\partial^2 u}{\partial x^2} + c\Phi(u) = f(x,t), \quad 0 < \alpha \leq 1 \quad (1)$$

with the initial condition:

$$u(x,0) = g(x) \quad (2)$$

where  $\partial^\alpha u / \partial t^\alpha$  denotes the local fractional derivative of  $u(x,t)$ ,  $a$ ,  $b$ , and  $c$  are parameters,  $f(x,t)$  and  $g(x)$  are given functions, and  $\Phi(u)$  is an appropriate non-linear function of  $u$ .

When the fractal dimension in eq. (1) is 1, eq. (1) occurs in various areas of applied mathematics, such as modeling of dynamics, heat conduction, and acoustic waves [21-39].

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### Mathematical fundamentals

In this section, we recall some definitions and properties of local fractional continuity, local fractional derivative, local fractional integral and Yang-Laplace transform of non-differential functions (for more details, see [1, 40]).

*Definition 1.* Assume the relation below exists [1, 40]:

$$|f(x) - f(x_0)| < \varepsilon^\alpha \quad (3)$$

with  $|x - x_0| < \delta$  for  $\varepsilon, \delta > 0$ . Then  $f(x)$  is local fractional continuous at  $x_0$  which is denoted by  $|\lim_{x \rightarrow x_0} f(x) = f(x_0)$ . If  $f(x)$  is local fractional continuous on the interval  $(a, b)$ , it is denoted by  $f(x) \in \mathcal{C}_\alpha(a, b)$ .

*Definition 2.* Let  $f(x) \in C_\alpha(a, b)$ . The local fractional derivative of  $f(x)$  of fractal order at the point  $x = x_0$  is given by [1-20]:

$$D_x^\alpha f(x_0) = \frac{d^\alpha}{dx^\alpha} f(x) \Big|_{x=x_0} = f^{(\alpha)}(x_0) = \lim_{x \rightarrow x_0} \frac{\Delta^\alpha [f(x) - f(x_0)]}{(x - x_0)^\alpha} \quad (4)$$

where  $\Delta[f(x) - f(x_0)] \cong \Gamma(\alpha + 1)[f(x) - f(x_0)]$ .

*Definition 3.* A partition of the interval  $[a, b]$  is denoted as  $(t_j, t_{j+1})$ ,  $j = 0, 1, \dots, N-1$ ,  $t_0 = a$  and  $t_N = b$  with  $\Delta t_j = t_{j+1} - t_j$  and  $\Delta t = \max\{\Delta t_0, \Delta t_1, \dots, \Delta t_N\}$ . The local fractional integral of  $f(x)$  in the interval  $[a, b]$  is given by [1-20]:

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1 + \alpha)} \int_a^b f(x) (dx)^\alpha = \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha \quad (5)$$

*Definition 4.* In the fractional space, the Mittag-Leffler function is given by [1, 40]:

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{(n\alpha)}}{\Gamma(1 + n\alpha)}, \quad 0 < \alpha \leq 1 \quad (6)$$

*Definition 5.* Let

$$\frac{1}{\Gamma(1 + \alpha)} \int_0^\infty |f(x)| (dx)^\alpha < k < \infty, \quad 0 < \alpha \leq 1$$

The Yang-Laplace transforms of  $f(x)$  is given by [1, 12, 21]:

$$L_\alpha \{f(x)\} = f_s^{L,\alpha}(s) = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha(-s^\alpha x^\alpha) f(x) (dx)^\alpha \quad (7)$$

where the latter integral converges and  $s^\alpha \in R^\alpha$ .

*Definition 6.* The inverse transform of the Yang-Laplace transforms of  $f(x)$  is given by [1, 12, 21]:

$$L_\alpha^{-1} \{f_s^{L,\alpha}(s)\} = f(x) = \frac{1}{(2\pi)^\alpha} \int_{\beta - i\infty}^{\beta + i\infty} E_\alpha(s^\alpha x^\alpha) f_s^{L,\alpha}(S) (ds)^\alpha \quad (8)$$

where  $s^\alpha = \beta^\alpha + i^\alpha \omega^\alpha$ ,  $\text{Re}(s) = \beta > 0$  and  $i^\alpha$  represents fractal imaginary unit [1, 12, 40].

Some useful formulas of local fractional derivative were given [1, 40]:

$$\frac{d^\alpha (x^{n\alpha})}{dx^\alpha} = \frac{\Gamma(1+n\alpha)x^{(n-1)\alpha}}{\Gamma[1+(n-1)\alpha]} \quad (9)$$

$$\frac{d^\alpha E_\alpha(x^\alpha)}{dx^\alpha} = E_\alpha(x^\alpha) \quad (10)$$

$$\frac{d^\alpha E_\alpha(\lambda x^\alpha)}{dx^\alpha} = \lambda E_\alpha(\lambda x^\alpha) \quad (11)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b E_\alpha(x^\alpha)(dx)^\alpha = E_\alpha(b^\alpha) - E_\alpha(a^\alpha) \quad (12)$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{n\alpha} (dx)^\alpha = \frac{\Gamma(1+n\alpha)[b^{(n+1)\alpha} - a^{(n+1)\alpha}]}{\Gamma[1+(n+1)\alpha]} \quad (13)$$

Next, we recall some basic properties of local fractional Yang-Laplace Transform [1]. Let  $L_\alpha\{f(x)\} = f_s^{L,\alpha}(s)$  and  $L_\alpha\{g(x)\} = g_s^{L,\alpha}(s)$ , then we have the following formulas:

$$L_\alpha\{af(x) + bg(x)\} = af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s) \quad (14)$$

$$L_\alpha\{E_\alpha(c^\alpha x^\alpha)f(x)\} = f_s^{L,\alpha}(s - c) \quad (15)$$

$$L_\alpha\{f^{(\alpha)}(x)\} = s^\alpha f_s^{L,\alpha}(s) - f(0) \quad (16)$$

$$L_\alpha\{E_\alpha(c^\alpha x^\alpha)\} = \frac{1}{s^\alpha - c^\alpha} \quad (17)$$

$$L_\alpha\{x^{k\alpha}\} = \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}} \quad (18)$$

### Local fractional Laplace decomposition method

The local fractional decomposition method [16-18] has been developed and applied to solve a class of local fractional PDE. Here, by using Yang-Laplace transform [1], we suggest a new analytical method.

Let us consider the following general non-linear local fractional PDE:

$$D_t^\alpha u(x,t) + R_\alpha u(x,t) + N_\alpha u(x,t) = f(x,t), \quad t > 0, \quad x \in R, \quad 0 < \alpha < 1 \quad (19)$$

where  $D_t^\alpha = \partial^\alpha / \partial t^\alpha$ ,  $R_\alpha$  denotes the linear local fractional differential operator,  $N_\alpha$  represents the general non-linear local fractional operator, and  $f(x,t)$  is a source term.

Taking local fractional Yang-Laplace transform on eq. (19), we obtain:

$$L_\alpha\{D_t^\alpha u(x,t)\} + L_\alpha\{R_\alpha u(x,t)\} + L_\alpha\{N_\alpha u(x,t)\} = L_\alpha\{f(x,t)\} \quad (20)$$

By applying the local fractional Yang-Laplace transform differentiation property (16), we have:

$$s^\alpha L_\alpha\{u(x,t)\} - u(x,0) = L_\alpha\{f(x,t)\} - L_\alpha\{R_\alpha u(x,t)\} - L_\alpha\{N_\alpha u(x,t)\} \quad (21)$$

or

$$L_\alpha \{u(x,t)\} = \frac{1}{s^\alpha} u(x,0) + \frac{1}{s^\alpha} L_\alpha \{f(x,t)\} - \frac{1}{s^\alpha} L_\alpha \{R_\alpha u(x,t)\} - \frac{1}{s^\alpha} L_\alpha \{N_\alpha u(x,t)\} \quad (22)$$

Operating with the Yang-Laplace inverse on both sides of eq. (22) gives:

$$u(x,t) = u(x,0) + L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [f(x,t)] \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [R_\alpha u(x,t)] \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [N_\alpha u(x,t)] \right\} \quad (23)$$

We are going to represent the solution in an infinite series given by:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \quad (24)$$

Substituting (24) into (23), we obtain:

$$\begin{aligned} \sum_{k=0}^{\infty} u_k(x,t) = & u(x,0) + L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [f(x,t)] \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left[ \sum_{k=0}^{\infty} R_\alpha u_k(x,t) \right] \right\} - \\ & - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha \left( \sum_{k=0}^{\infty} \tilde{A}_k \right) \right\} \end{aligned} \quad (25)$$

where  $\tilde{A}_k$  are the Adomian polynomials of non-linear operator  $N_\alpha(u)$ .

The first few polynomials are given by:

$$\begin{aligned} \tilde{A}_0 &= N_\alpha(u_0) \\ \tilde{A}_1 &= u_1 N'_\alpha(u_0) \\ \tilde{A}_2 &= u_2 N'_\alpha(u_0) + \frac{1}{2!} u_1^2 N''_\alpha(u_0) \end{aligned} \quad (26)$$

$$\tilde{A}_3 = u_3 N'_\alpha(u_0) + u_1 u_2 N''_\alpha(u_0) + \frac{1}{3!} u_1^3 N'''_\alpha(u_0)$$

Comparing the left and right hand sides of (25), we have:

$$\begin{aligned} u_0(x,t) &= u(x,0) + L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [f(x,t)] \right\} \\ u_1 &= -L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (R_\alpha u_0) \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (\tilde{A}_0) \right\} \\ u_2 &= -L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (R_\alpha u_1) \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (\tilde{A}_1) \right\} \end{aligned} \quad (27)$$

and so on.

The recursive relation in its general form is:

$$\begin{aligned} u_0(x,t) &= u(x,0) + L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [f(x,t)] \right\} \\ u_{n+1} &= -L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (R_\alpha u_n) \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (\tilde{A}_n) \right\} \end{aligned} \quad (28)$$

Notice that the formulas are only valid for non-differentiable functions.

**Applications**

In this section, we present the solutions of non-linear fractional Burgers eq. (1) by an application of local fractional Yang-Laplace decomposition method.

For eq. (1), we have:

$$R_\alpha u = -b \frac{\partial^2 u}{\partial x^2}$$

$$N_\alpha u = au \frac{\partial u}{\partial x} + c\Phi(u)$$

Then, by eq. (28), we obtain:

$$u_0(x,t) = g(x) + L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha [f(x,t)] \right\}$$

$$u_{n+1} = -L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (R_\alpha u_n) \right\} - L_\alpha^{-1} \left\{ \frac{1}{s^\alpha} L_\alpha (\tilde{B}_n) \right\}, \quad (n = 0, 1, 2, \dots)$$

where

$$\tilde{B}_0 = au_0 \frac{\partial u_0}{\partial x} + c\Phi(u_0)$$

$$\tilde{B}_1 = a \left( u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} \right) + cu_1 \Phi'(u_0)$$

$$\tilde{B}_2 = a \left( u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x} \right) + cu_2 \Phi'(u_0) + \frac{c}{2!} u_1^2 \Phi''(u_0)$$

$$\tilde{B}_3 = a \left( u_3 \frac{\partial u_0}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_3}{\partial x} \right) + cu_3 \Phi'(u_0) + cu_1 u_2 \Phi''(u_0) + \frac{c}{3!} u_1^3 \Phi'''(u_0)$$

and so on.

Thus the *n-term* approximate solution of eq. (1) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots + u_n(x,t)$$

For example, we consider eq. (1) in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + u = 0, \quad 0 < \alpha \leq 1$$

with the initial condition:

$$u(x,0) = x$$

By recursive relations (29), we obtain:

$$u_0(x,t) = x$$

$$u_1(x,t) = \frac{-2xt^\alpha}{\Gamma(1+\alpha)}$$

$$u_2(x,t) = \frac{6xt^{2\alpha}}{\Gamma(1+2\alpha)}$$

$$u_3(x,t) = -\frac{4\Gamma(1+2\alpha) + 18\Gamma^2(1+\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)} xt^{3\alpha}$$

$$u_4(x,t) = \frac{12\Gamma^2(1+2\alpha) + 54\Gamma^2(1+\alpha)\Gamma(1+2\alpha) + 24\Gamma(1+\alpha)\Gamma(1+3\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} xt^{4\alpha}$$

and so on.

Thus, the solution of (30) is given:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \dots$$

For  $\alpha = 1$ , we have:

$$u(x,t) = x - 2xt + 3xt^2 - \frac{13}{3}xt^3 + \frac{25}{4}xt^4 + \dots$$

The series closed to the exact solution is:

$$u(x,t) = \frac{x}{2e^t - 1}$$

## Conclusion

We presented the application of local fractional Yang-Laplace decomposition method to a local fractional Burgers equation. It may be concluded that the local fractional Yang-Laplace decomposition method is very efficient in finding the analytical solutions for a wide class of non-linear local fractional differential equations. Our results show that the method gives high accuracy series solutions that converge very rapidly. The useful method may considerably benefit scientists working in the field of non-linear PDE.

## Nomenclature

$t$  – time co-ordinate, [s]  
 $x$  – space co-ordinate, [m]

*Greek symbols*

$\alpha$  – fractal order, [-]  
 $\partial^\alpha/\partial t^\alpha$  – local fractional derivative, [-]

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