SHALLOW WATER WAVES IN POROUS MEDIUM FOR COAST PROTECTION

by

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This paper extends the Hirota-Satsuma equation in continuum mechanics to its fractional partner in fractal porous media in shallow water for absorbing wave energy and preventing tsunami. Its derivation is briefly introduced using the fractional momentum law and He's fractional derivative. The fractional complex transform is adopted to elucidate its basic solution properties, and a modification of the exp-function method is used to solve the equation. The paper concludes that the kinetic energy of the travelling wave tends to be vanished when the value of the fractional order is less than one.

Key words: *He's fractional derivative, fractional differential equations, generalized exponential rational function method*

Introduction

Coast protection is defense against solitary waves like tsunami by construction of a 3-D porous structure in near-coast waters. It is considered that such shallow water waves can be described by fractional calculus [1, 2], which becomes a powerful tool to modeling phenomena for discontinuous or fractal media [3-10]. A heuristic explanation of fractional calculus is given in [11-13], for examples, water is continuous on any macro scales, and the classical mechanical laws should be followed. However, when the water is observed in molecular scales, discontinuous water molecules and other ions involved can be revealed. In such a case, all continuum mechanical laws are forbidden, and their fractional partners have to be adopted to explain, for example, the effect of molecule size on the diffusion of ions or inks in water [11, 14]. It is well-known that the diffusion process can be described by Fick's laws in continuum mechanics, that means the fractal space-time in an extremely small scale (*e. g.*, few nanoscales) can be converted in an approximate continuous one at some a larger scale, this is the very assumption of the fractional complex transform in fractional calculus [15-17], which is to convert a fractional space-time on a small scale into its continuous one on a larger scale. This can be understood by looking at a Google Map. At a small scale, we can see many little rivers or lakes in a city, when we enlarge the scale, all rivers or lakes disappear, and the discontinuous land of the city on the small scale becomes continuous on a larger scale.

In this paper, we will study a fractional modification of the well-known Hirota-Satsuma equation using He's fractional derivative [1, 2, 12, 18-23], which is defined:

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$$\mathrm{D}_t^\alpha u = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}[u_0(s)-u(s)]\mathrm{d}s \tag{1}$$

where $u_0(s)$ is a known function. This definition has been widely used in engineering to explain hidden phenomena in polar bears, wools and cocoons [18-24].

The Hirota-Satsuma equation is to describe shallow water waves, which was first derived by Hirota and Satsuma [25], and its solitary properties have been widely studied in [26, 27]. The shallow water wave is of great importance in the coast protection. In this paper a 3-D porous structure is suggested to prevent solitary waves in coastal waters.

Hirota-Satsuma equations and their fractional partner

It is well-known that the shallow water waves can be described by Hirota-Satsuma equations [25]:

$$\begin{cases} \varphi_t - 0.5\varphi_{xxx} + 3\varphi\varphi_x - 3(\psi w)_x = 0 \\ \psi_t + \psi_{xxx} - 3\varphi\psi_x = 0 \\ w_t + w_{xxx} - 3\varphi w_x = 0 \end{cases} \tag{2}$$

and its fractional partner describes shallow water in porous medium, which is used to absorb wave energy and prevent tsunami:

$$\begin{cases} \mathrm{D}_t^\alpha\varphi - \dfrac{1}{2}\mathrm{D}_x^{3\alpha}\varphi + 3\varphi\mathrm{D}_x^\alpha\varphi - 3\mathrm{D}_x^\alpha(\psi w) = 0 \\ \mathrm{D}_t^\alpha\psi + \mathrm{D}_x^{3\alpha}\psi - 3\varphi\mathrm{D}_x^\alpha\psi = 0 \\ \mathrm{D}_t^\alpha w + \mathrm{D}_x^{3\alpha}w - 3\varphi\mathrm{D}_x^\alpha w = 0 \end{cases} \tag{3}$$

where $0<\alpha\le 1$, $\varphi=\varphi(x,t)$, $\psi=\psi(x,t)$, $w=w(x,t)$ are wave velocities in three orthogonal directions. In eq. (3) a fractional law for momentum equation reads [12]:

$$\rho\frac{\mathrm{D}^\alpha\vec{\mathrm{u}}}{\mathrm{D}t^\alpha} = \rho\left(\frac{\partial^\alpha\vec{\mathrm{u}}}{\partial t^\alpha} + \varphi\frac{\partial^\alpha\vec{\mathrm{u}}}{\partial x^\alpha} + \psi\frac{\partial^\alpha\vec{\mathrm{u}}}{\partial y^\alpha} + w\frac{\partial^\alpha\vec{\mathrm{u}}}{\partial z^\alpha}\right) = -\nabla^\alpha p + \nabla^\alpha\boldsymbol{\tau} + \rho\mathrm{g} \tag{4}$$

where ρ is the density, $\vec{\mathrm{u}}=(\varphi,\psi,w)$ – the flow velocity, p – the pressure, $\boldsymbol{\tau}$ – the deviatoric stress tensor, g represents gravity, and ∇^α is the fractional del operator defined:

$$\nabla^\alpha = \frac{\partial^\alpha}{\partial x^\alpha}\vec{\mathrm{i}} + \frac{\partial^\alpha}{\partial y^\alpha}\vec{\mathrm{j}} + \frac{\partial^\alpha}{\partial z^\alpha}\vec{\mathrm{k}} \tag{5}$$

where $\mathrm{D}^\alpha/\mathrm{D}t^\alpha$ is the fractional material derivative defined:

$$\frac{\mathrm{D}^\alpha}{\mathrm{D}t^\alpha} = \frac{\partial^\alpha}{\partial t^\alpha} + \varphi\frac{\partial^\alpha}{\partial x^\alpha} + \psi\frac{\partial^\alpha}{\partial y^\alpha} + w\frac{\partial^\alpha}{\partial z^\alpha} \tag{6}$$

When $\alpha=1$, eq. (4) becomes the well-known momentum equation in fluid mechanics:

$$\rho\left[\frac{\partial\vec{\mathrm{u}}}{\partial t} + \varphi\frac{\partial\vec{\mathrm{u}}}{\partial x} + \psi\frac{\partial\vec{\mathrm{u}}}{\partial y} + w\frac{\partial\vec{\mathrm{u}}}{\partial z}\right] = -\nabla p + \nabla\boldsymbol{\tau} + \rho\mathrm{g} \tag{7}$$

Solitary solutions of fractional Hirota-Satsuma equations

There are many analytical methods to solve fractional differential equations, for examples, the homotopy perturbation method [28, 29], variational iteration method [30-32], Adomaian decomposition method [33], sub-equation method [34-36], homotopy analysis method [37], G'/G method [38], and others [39], among which the exp-function method [40-43] seems to be more suitable for fractional calculus, and its modification, the generalized exponential rational function method [44], has also been caught much attention recently.

To solve eq. (3), we give the fractional complex transformation [15-17]:

$$\begin{cases} \varphi(x,t) = \Phi(\xi) \\ \psi(x,t) = \Psi(\xi) \\ w(x,t) = W(\xi) \\ \xi = \dfrac{kx^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{ct^{\alpha}}{\Gamma(\alpha+1)} \end{cases} \tag{8}$$

where k and c are constants, such that eq. (3) takes the following form:

$$\begin{cases} c\Phi' - \dfrac{1}{2}k^3\Phi''' + 3k\Phi\Phi' - 3k(\Psi' W + W'\Psi) = 0 \\ c\Psi' + k^3\Psi''' - 3k\Phi\Psi' = 0 \\ cW' + k^3 W''' - 3k\Phi W' = 0 \end{cases} \tag{9}$$

The fractional complex transform [15-17] is an approximate transform of a fractal space or space-time into its continuous partner. A geometrical explanation was given in [13, 45]. To explain the transform, we consider a television screen, which is smooth at any observable scales, however, it consists of matrix points on the surface, which can be considered a hierarchical level of a fractal structure, while a smooth TV screen is observed on a larger hierarchical level. For our problem, when we observe the porous medium on a scale of porous size, water becomes discontinuous. On the other hand, when we observe a shallow solitary wave at a coast on a scale of, for example, 10 kilometers, the water becomes approximately continuous. That means water is discontinuous on scale of x, and it becomes approximately continuous on scale of x^a, this is the geometrical properties of the fractional complex transform [13, 45].

We suppose that the exact solution of eq. (9) can be expressed:

$$\begin{cases} \Phi(\xi) = \displaystyle\sum_{n=0}^{N_1} \frac{p_n}{(1 \pm a^{\xi})^n}, \\ \Psi(\xi) = \displaystyle\sum_{m=0}^{M_1} \frac{q_m}{(1 \pm a^{\xi})^m}, \\ W(\xi) = \displaystyle\sum_{l=0}^{L_1} \frac{r_l}{(1 \pm a^{\xi})^l} \end{cases} \tag{10}$$

where $p_n(n = 0,1,\cdots,N_1)$, $q_m(m = 0,1,\cdots,M_1)$, and $r_l(l = 0,1,\cdots,L_1)$ are constants to be determined later.

Equation (10) is constructed according to the exp-function method [40-43], and it was called as the generalized exponential rational function method in [44].

By balancing the highest order derivative term and non-linear term in eq. (9), we receive $N_1+3=2N_1+1, N_1+3=M_1+L_1+1, M_1+3=N_1+M_1+1,$ and $L_1+3=N_1+L_1+1.$

If we choose $N_1=2, M_1=2,$ and $L_1=2,$ then eq. (10) becomes:

$$\begin{cases}\Phi(\xi)=p_0+\dfrac{p_1}{1\pm a^\xi}+\dfrac{p_2}{(1\pm a^\xi)^2}\\ \Psi(\xi)=q_0+\dfrac{q_1}{1\pm a^\xi}+\dfrac{q_2}{(1\pm a^\xi)^2}\\ W(\xi)=r_0+\dfrac{r_1}{1\pm a^\xi}+\dfrac{r_2}{(1\pm a^\xi)^2}\end{cases} \tag{11}$$

Substituting eq. (11) into eq. (9) and then equating each coefficient of the same powers of a^ξ to zero, we get a series of algebraic equations with respect to $p_0, p_1, p_2, q_0, q_1, q_2, r_0, r_1,$ and r_2. Solving the algebraic equations by some a mathematical software, we have the following results:

Case 1

$$p_0=\frac{1}{3}\frac{c+k^3(\ln a)^2}{k},\quad p_1=-4k^2(\ln a)^2,\quad p_2=4k^2(\ln a)^2$$

$$q_0=-\frac{2}{3}\frac{[4cr_1+k^3(\ln a)^2r_1+6k^3(\ln a)^2r_0]k(\ln a)^2}{r_1{}^2},\quad q_1=\frac{4k^4(\ln a)^4}{r_1},\quad q_2=-\frac{4k^4(\ln a)^4}{r_1}$$

$$r_0=r_0,\quad r_1=r_1,\quad r_2=-r_1$$

Case 2

$$p_0=\frac{1}{3}\frac{c+k^3(\ln a)^2}{k},\quad p_1=-2k^2(\ln a)^2,\quad p_2=2k^2(\ln a)^2$$

$$q_0=-\frac{1}{3}\frac{[4cr_0+4cr_1+k^3(\ln a)^2r_0+k^3(\ln a)^2r_1]k(\ln a)^2}{r_1{}^2},\quad q_1=\frac{1}{3}\frac{k(\ln a)^2[4c+k^3(\ln a)^2]}{r_1}$$

$$q_2=0,\quad r_0=r_0,\quad r_1=r_1,\quad r_2=0$$

Case 3

$$p_0=\frac{1}{6}\frac{-2c+k^3(\ln a)^2}{k},\quad p_1=-2k^2(\ln a)^2,\quad p_2=2k^2(\ln a)^2$$

$$q_0=q_0,\quad q_1=0,\quad q_2=0$$

$$r_0=r_0,\quad r_1=0,\quad r_2=0$$

Hereby we consider the Case 3, which allows the following solitary wave:

$$\begin{cases}\varphi(x,t)=\dfrac{1}{6}\dfrac{-2c+k^3(\ln a)^2}{k}-\dfrac{2k^2(\ln a)^2}{1-a^\xi}+\dfrac{2k^2(\ln a)^2}{(1-a^\xi)^2}\\ \psi(x,t)=q_0\\ w(x,t)=r_0\end{cases} \tag{12}$$

where $\xi=kx^\alpha/\Gamma(\alpha+1)+ct^\alpha/\Gamma(\alpha+1).$

Equation (12) shows that the wave surface is an almost flat one except one or multiple peak. Such phenomenon implies that the wave energy is focused on the peak wave, and the peak height and wave velocity can be controlled by the fractional order, which is directly relative to the fractal dimensions of the porous structure [46].

Discussion and conclusion

In order to elucidate basic solution properties of the fractional Hirota-Satsuma equation, we consider a simple solution of a sine wave for φ:

$$\varphi(x,t) = A\sin(\omega\xi) = A\sin\left[\frac{\omega}{\Gamma(\alpha+1)}(kx^{\alpha}+ct^{\alpha})\right] \tag{13}$$

where A is the amplitude, and ω is the frequency. The partial differentiation of φ with respect to time is:

$$\frac{\partial\varphi(x,t)}{\partial t} = \frac{cA\omega}{\Gamma(\alpha+1)}t^{\alpha-1}\cos\left[\frac{\omega}{\Gamma(\alpha+1)}(kx^{\alpha}+ct^{\alpha})\right] \tag{14}$$

When $\alpha = 1$ it becomes a cosine function. But when $\alpha < 1$ for a porous medium, $\partial\varphi/\partial t$ tends to an infinite small value when t tends to an infinite large one, and its kinetic energy is vanished completely when time tends to infinity, this means the sine wave is disappeared. This energy is dissipative due to the porous medium, In [47] an effective way is suggested to construct porous structure using flexible balloons, and the value of the fractional order can be adjusted by controlling the size and number of balloons, as a result, the direct and morphology of the solitary waves can be effectively controlled.

To be concluded, we show that shallow water waves in porous media can be expressed by the fractional Hirota-Satsuma equations, its wave morphology can effectively be controlled by the fractional order. As pointed out by Wu and Liang [46] that the value of the fractional derivative depends on the fractal dimensions of the porous medium, that means we can adjust the porous structure to absorb wave energy and prevent tsunami.

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Nomenclature

t – time co-ordinate, [s]
x – space co-ordinate, [m]

Greek symbols

α – fractional order, [–]

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