NEW MULTI-SOLITON SOLUTIONS OF WHITHAM-BROER-KAUP SHALLOW-WATER-WAVE EQUATIONS

by

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In this paper, new and more general Whitham-Broer-Kaup equations which can describe the propagation of shallow-water waves are exactly solved in the framework of Hirota's bilinear method and new multi-soliton solutions are obtained. To be specific, the Whitham-Broer-Kaup equations are first reduced into Ablowitz-Kaup-Newell-Segur equations. With the help of this equations, bilinear forms of the Whitham-Broer-Kaup equations are then derived. Based on the derived bilinear forms, new one-soliton solutions, two-soliton solutions, three-soliton solutions, and the uniform formulae of n-soliton solutions are finally obtained. It is shown that adopting the bilinear forms without loss of generality play a key role in obtaining these new multi-soliton solutions.

Key words: *Whitham-Broer-Kaup equations, Ablowitz-Kaup-Newell-Segur equations, Hirota's bilinear method, Bilinear forms, soliton solution*

Introduction

Non-linear PDE are often used to describe some non-linear phenomena of the real world involved in many fields from physics to biology, economics, chemistry, mechanics, fluid dynamics, engineering, *etc.* Usually, researchers resort to solutions of such non-linear PDE for more insight into these physical phenomena. Soliton is such a kind of non-linear phenomenon which not only can be observed in nature, but also can be produced through experiment. As pointed out by Drazin and Johnson [1], it is not easy to give a comprehensive and precise definition of a soliton. However, one can associate the term with any solution of non-linear PDE which: represents a wave of permanent form, is localized, so that it decays or approaches a constant at infinity, and can undergo a strong interaction with other solitons preserving its identity. With the development of soliton theory, finding soliton solutions [2-6] of non-linear PDE has become one of the most exciting and extremely active areas of research.

In 1971, Hirota proposed a direct method [7] for constructing multi-soliton solutions of non-linear PDE. Since put forward by Hirota, Hirota's bilinear method has developed to a systematic method [8] for multi-soliton solutions [9-19]. In this paper, we shall extend Hirota's bilinear method to new and more general Whitham-Broer-Kaup (WBK) equations with arbitrary constant coefficients γ_i $(i=1,2,\ldots,6)$ [20]:

$$u_t+\gamma_1 uu_x+\gamma_2 v_x+\gamma_3 u_{xx}=0 \tag{1}$$

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$$v_t + \gamma_4 u_x v + \gamma_4 u v_x - \gamma_5 v_{xx} + \gamma_6 u_{xxx} = 0 \tag{2}$$

for constructing new multi-soliton solutions. It should be noted that eqs. (1) and (2) are more general than the following known WBK model for the dispersive long waves in shallow water [17-19]:

$$u_t + u u_x + v_x + \gamma u_{xx} = 0 \tag{3}$$

$$v_t + (uv)_x + \beta u_{xxx} - \gamma v_{xx} = 0 \tag{4}$$

Besides, if we select appropriate values of γ_i $(i = 1, 2, \ldots, 6)$ then eqs. (1) and (2) give some other known non-linear PDE, such as the approximate equations for long water waves [21], the Boussinesq-Burgers equations [22]. In [20], some symmetries and similarity reductions of eqs. (1) and (2) are obtained. Recently, eqs. (3) and (4) have attached much attention and many exact solutions like those in [23-25] have been constructed. It is worth mentioning that Lin *et al.* [17, 18] and Wang *et al.* [19] obtained multi-soliton solutions in terms of double Wronskian determinant. As far as we know, there are no multi-soliton solutions and other solutions of eqs. (1) and (2) have been reported in literature.

Bilinear forms

In order to derive the bilinear forms conveniently, we reduce eqs. (1) and (2) in advance.

Theorem 1. If let

$$u = a\frac{A_x}{A}, \quad v = -a^2\frac{\gamma_1}{\gamma_2}AB + a\frac{a\gamma_1 - 2\gamma_3}{2\gamma_2}\left(-\frac{A_x^2}{A^2} + \frac{A_{xx}}{A}\right) \tag{5}$$

where a is an arbitrary constant, A and B are undetermined smooth formations of x and t, then the WBK eqs. (1) and (2) reduce into the AKNS equations:

$$A_t - \frac{1}{2}a\gamma_1(2A^2B - A_{xx}) = 0, \quad B_t - \frac{1}{2}a\gamma_1(-2A^2B + A_{xx}) = 0 \tag{6}$$

under the constraints:

$$\gamma_4 = \gamma_1, \quad \gamma_5 = \gamma_3, \quad \gamma_6 = \frac{a^2\gamma_1^2}{4\gamma_2} \tag{7}$$

Proof. Supposing that:

$$u = a(\ln A)_x, \quad v = b(\ln A)_{xx} + cAB \tag{8}$$

where a, b, and c are constants to be determined, and then substituting eqs. (8) into eqs. (1) and (2), we arrive at eqs. (6) under the constraints (7). We finish the proof of Theorem 1.

Theorem 2. Under the constraints (7), the WBK eqs. (1) and 2 possess the bilinear forms:

$$\mathrm{D}_t g f = \frac{1}{2}a\gamma_1\left[-\mathrm{D}_x^2 g f + \frac{g}{f}(\mathrm{D}_x^2 f f + 2gh)\right] \tag{9}$$

$$\mathrm{D}_t h f = \frac{1}{2}a\gamma_1\left[\mathrm{D}_x^2 h f - \frac{h}{f}(\mathrm{D}_x^2 f f + 2gh)\right] \tag{10}$$

where $f=f(x,t)$, $g=g(x,t)$, $h=h(x,t)$, D_x and D_t are Hirota's differential operators [8].

Proof. Starting from eqs. (6), we suppose that:

$$A=\frac{g}{f},\quad B=\frac{h}{f} \tag{11}$$

Using Hirota's bilinear differential operators and eqs. (11), we can re-write eqs. (6) as eqs. (9) and (10). Thus, the proof of Theorem 2 is end.

Multi-soliton solutions

Generally speaking, it is difficulty in using the bilinear forms (9) and (10) to construct multi-soliton solutions of eqs. (1) and (2). Usually one assumes $\mathrm{D}_x^2 ff+2gh=0$ to use a special case [13, 16-19] of eqs. (9) and (10) for the multi-soliton solutions. This is not the starting point of this paper. Without loss of generality, we shall construct new multi-soliton solutions by employing the bilinear forms (9) and (10) with $\mathrm{D}_x^2 ff+2gh\neq 0$.

Theorem 3. Under the constraints (7), the WBK eqs. (1) and (2) possesses the uniform formulae of *n*-soliton solutions determined by:

$$u=a\frac{g_{nx}f_n-f_{nx}g_n}{f_ng_n},\quad v=-a^2\frac{\gamma_1}{\gamma_2}\frac{g_nh_n}{f_n^2}+a\frac{a\gamma_1-2\gamma_3}{2\gamma_2}\left(\frac{f_{nx}^2}{f_n^2}-\frac{g_{nx}^2}{g_n^2}-\frac{f_{nxx}}{f_n}+\frac{g_{nxx}}{g_n}\right) \tag{12}$$

with

$$g_n=\frac{\alpha}{\sqrt{2}}\mathrm{e}^{\alpha^2 t}\sum_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j(\xi_j+2\theta_j+\ln 2)+\sum_{1\le j<l}^{n}\mu_j\mu_l A_{jl}} \tag{13}$$

$$h_n=\frac{\alpha}{\sqrt{2}}\mathrm{e}^{-\alpha^2 t}\sum_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j\left(\xi_j-2\theta_j+\ln 2\right)+\sum_{1\le j<l}^{n}\mu_j\mu_l A_{jl}},\quad f_n=\sum_{\mu=0,1}\mathrm{e}^{\sum_{j=1}^{n}\mu_j\left(\xi_j+\ln 2\right)+\sum_{1\le j<l}^{n}\mu_j\mu_l A_{jl}} \tag{14}$$

$$\xi_j=\omega_j t+k_j x+\xi_j^0,\quad k_j^2=-4\alpha^2\sinh^2\theta_j,\quad \omega_j=\frac{1}{2}a\gamma_1\alpha^2\sinh 2\theta_j,\quad (j=1,2,\dots,n) \tag{15}$$

$$\mathrm{e}^{A_{jl}}=\frac{\sinh^2\dfrac{\theta_j-\theta_l}{2}}{\sinh^2\dfrac{\theta_j+\theta_l}{2}},\quad (1\le j<l\le n) \tag{16}$$

where α is a constant parameter and ξ_i^0 – an arbitrary constant. The summation $\Sigma_{\mu=0,1}$ refers to all possible combinations of each $\mu_i=0,1$ for $i=1,2,\dots,n$.

Proof. We first introduce a parameter α which is independent with *x* and *t* so that:

$$\mathrm{D}_x^2 ff+2gh=a^2 f^2 \tag{17}$$

then the bilinear forms (9) and (10) become:

$$\left(\mathrm{D}_t+\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)gf=\alpha^2 gh,\quad \left(\mathrm{D}_t-\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)hf=-\alpha^2 hf \tag{18}$$

Further taking the transformations [9]:

$$f=\overline{f}\,,\quad g=\mathrm{e}^{\alpha^2 t}\overline{g}\,,\quad h=\mathrm{e}^{-\alpha^2 t}\overline{h} \tag{19}$$

and using eqs. (18) and (19), we convert eqs. (9) and (10) into the bilinear forms of $\overline{f}$, $\overline{g}$, and $\overline{h}$ (here we still write them as f, g, and h for convenience):

$$\left(\mathrm{D}_t+\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)gf=0\,,\quad \left(\mathrm{D}_t-\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)hf=0\,,\quad \mathrm{D}_x^2 ff=-2gh+a^2f^2 \tag{20}$$

In what follows, using eqs. (20) we construct multi-soliton solutions of eqs. (1) and (2). To construct one-soliton solutions, we introduce a parameter ε and expand f, g, and h:

$$f=1+\varepsilon f^{(1)}+\varepsilon^2 f^{(2)}+\cdots+\varepsilon^j f^{(j)}+\cdots \tag{21}$$

$$g=g^{(0)}+\varepsilon g^{(1)}+\varepsilon^2 g^{(2)}+\cdots+\varepsilon^j g^{(j)}+\cdots,\quad h=h^{(0)}+\varepsilon h^{(1)}+\varepsilon^2 h^{(2)}+\cdots+\varepsilon^j h^{(j)}+\cdots \tag{22}$$

Substituting eqs. (21) and (22) into eqs. (20) and then collecting all the coefficients with same order of ε, we get a system of differential equations (SDE):

$$g_t^{(0)}+\frac{1}{2}a\gamma_1 g_{xx}^{(0)}=0\,,\quad h_t^{(0)}-\frac{1}{2}a\gamma_1 h_{xx}^{(0)}=0\,,\quad 2g^{(0)}h^{(0)}=\alpha^2 \tag{23}$$

$$g_t^{(1)}+\frac{1}{2}a\gamma_1 g_{xx}^{(1)}=-\left(\mathrm{D}_t+\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)g^{(0)}f^{(1)} \tag{24}$$

$$h_t^{(1)}-\frac{1}{2}a\gamma_1 h_{xx}^{(1)}=-\left(\mathrm{D}_t-\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)h^{(0)}f^{(1)}\,,\quad f_{xx}^{(1)}=-g^{(0)}h^{(1)}-g^{(1)}h^{(1)}+\alpha^2 f^{(1)} \tag{25}$$

$$g_t^{(2)}+\frac{1}{2}a\gamma_1 g_{xx}^{(2)}=-\left(\mathrm{D}_t+\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)\left(g^{(0)}f^{(2)}+g^{(1)}f^{(1)}\right) \tag{26}$$

$$h_t^{(2)}+\frac{1}{2}a\gamma_1 h_{xx}^{(2)}=-\left(\mathrm{D}_t-\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)\left(h^{(0)}f^{(2)}+h^{(1)}f^{(1)}\right) \tag{27}$$

$$f_{xx}^{(2)}=-\frac{1}{2}\mathrm{D}_x^2 f^{(1)}f^{(1)}-g^{(0)}h^{(2)}-g^{(1)}h^{(1)}-g^{(2)}h^{(0)}+\alpha^2\left[f^{(2)}+\frac{1}{2}\left(f^{(1)}\right)^2\right] \tag{28}$$

$$g_t^{(3)}+\frac{1}{2}a\gamma_1 g_{xx}^{(3)}=-\left(\mathrm{D}_t+\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)\left(g^{(0)}f^{(3)}+g^{(1)}f^{(2)}+g^{(2)}f^{(1)}\right) \tag{29}$$

$$h_t^{(3)}+\frac{1}{2}a\gamma_1 h_{xx}^{(3)}=-\left(\mathrm{D}_t-\frac{1}{2}a\gamma_1\mathrm{D}_x^2\right)\left(h^{(0)}f^{(3)}+h^{(1)}f^{(2)}+h^{(2)}f^{(1)}\right) \tag{30}$$

$$f_{xx}^{(3)}=-D_x^2 f^{(1)}f^{(2)}-g^{(0)}h^{(3)}-g^{(1)}h^{(2)}-g^{(2)}h^{(1)}-g^{(3)}h^{(0)}+\alpha^2\left(f^{(3)}+f^{(1)}f^{(2)}\right) \tag{31}$$

and so forth.

From eqs. (23) we have:

$$g^{(0)}=h^{(0)}=\frac{\alpha}{\sqrt{2}} \tag{32}$$

Substituting eq. (32) into eqs. (24) and (25), we can see that:

$$f^{(1)} = 2e^{\xi_1}, \quad g^{(1)} = \sqrt{2}\alpha e^{\xi_1+2\theta_1}, \quad h^{(1)} = \sqrt{2}\alpha e^{\xi_1-2\theta_1} \tag{33}$$

$$\xi_1 = \omega_1 t + k_1 x + \xi_1^{(0)}, \quad k_1^2 = -2\alpha^2 \sinh^2\theta_1, \quad \omega_1 = \frac{1}{2}a\gamma_1\alpha^2 \sinh 2\theta_1 \tag{34}$$

satisfy eqs. (24) and (25).

If $g^{(2)} = g^{(3)} = h^{(2)} = h^{(3)} = f^{(2)} = f^{(3)} = \cdots = 0$, then eqs. (33) and (34) satisfy all the other equations in previous SDE. Thus, eqs. (21) and (22) are truncated. Letting $\varepsilon = 1$ yields:

$$f_1 = 1 + 2e^{\xi_1}, \quad g_1 = \frac{\alpha}{\sqrt{2}}\left(1 + 2e^{\xi_1+2\theta_1}\right), \quad h_1 = \frac{\alpha}{\sqrt{2}}\left(1 + 2e^{\xi_1-2\theta_1}\right) \tag{35}$$

Using eqs. (5), (6), (11), (19), and (35), we obtain one-soliton solutions of eqs. (1) and (2):

$$u = a\frac{g_{1x}f_1 - f_{1x}g_1}{f_1 g_1}, \quad v = -a^2\frac{\gamma_1}{\gamma_2}\frac{g_1 h_1}{f_1^2} + a\frac{a\gamma_1 - 2\gamma_3}{2\gamma_2}\left(\frac{f_{1x}^2}{f_1^2} - \frac{g_{1x}^2}{g_1^2} - \frac{f_{1xx}}{f_1} + \frac{g_{1xx}}{g_1}\right) \tag{36}$$

To construct two-soliton solutions of eqs. (1) and (2), we select:

$$f^{(1)} = 2(e^{\xi_1} + e^{\xi_2}), \quad g^{(1)} = \sqrt{2}\alpha\left(e^{\xi_1+2\theta_1} + e^{\xi_2+2\theta_2}\right), \quad h^{(1)} = \sqrt{2}\alpha\left(e^{\xi_1-2\theta_1} + e^{\xi_2-2\theta_2}\right) \tag{37}$$

and suppose that:

$$\left(D_t + \frac{1}{2}a\gamma_1 D_x^2\right)\left(g^{(1)}f^{(2)} + g^{(2)}f^{(1)}\right) = 0 \tag{38}$$

$$\left(D_t - \frac{1}{2}a\gamma_1 D_x^2\right)\left(h^{(1)}f^{(2)} + h^{(2)}f^{(1)}\right) = 0, \quad D_x^2 f^{(1)}f^{(2)} + g^{(1)}h^{(2)} + g^{(2)}h^{(1)} - \alpha^2 f^{(1)}f^{(2)} = 0 \tag{39}$$

It is easy to see that eqs. (26)-(28), (38), and (39) have solutions:

$$f^{(2)} = 4e^{\xi_1+\xi_2+A_{12}}, \quad g^{(2)} = 2\sqrt{2}\alpha e^{\xi_1+\xi_2+2\theta_1+2\theta_2+A_{12}}, \quad h^{(2)} = 2\sqrt{2}\alpha e^{\xi_1+\xi_2-2\theta_1-2\theta_2+A_{12}} \tag{40}$$

Substituting eqs. (37), (40), and (41) into eqs. (29)-(31), we have:

$$f^{(3)} = g^{(3)} = h^{(3)} = f^{(4)} = g^{(4)} = h^{(4)} = \cdots = 0 \tag{41}$$

In this case, eqs. (17) and (18) have solutions:

$$f_2 = 1 + 2(e^{\xi_1} + e^{\xi_2}) + 4e^{\xi_1+\xi_2+A_{12}}, \quad g_2 = \frac{\alpha}{\sqrt{2}}e^{\alpha^2 t}\left[1 + 2(e^{\xi_1+2\theta_1} + e^{\xi_2+2\theta_2}) + 4e^{\xi_1+\xi_2+2\theta_1+2\theta_2+A_{12}}\right] \tag{42}$$

$$h_2 = \frac{\alpha}{\sqrt{2}}e^{-\alpha^2 t}\left[1 + 2(e^{\xi_1-2\theta_1} + e^{\xi_2-2\theta_2}) + 4e^{\xi_1+\xi_2-2\theta_1-2\theta_2+A_{12}}\right] \tag{43}$$

We therefore obtain two-soliton solutions of eqs. (1) and (2):

$$u = a\frac{g_{2x}f_2 - f_{2x}g_2}{f_2 g_2}, \quad v = -a^2\frac{\gamma_1}{\gamma_2}\frac{g_2 h_2}{f_2^2} + a\frac{a\gamma_1 - 2\gamma_3}{2\gamma_2}\left(\frac{f_{2x}^2}{f_2^2} - \frac{g_{2x}^2}{g_2^2} - \frac{f_{2xx}}{f_2} + \frac{g_{2xx}}{g_2}\right) \tag{44}$$

Similiarly, three-soliton solutions of eqs. (1) and (2) are obtained:

$$u = a\frac{g_{3x}f_3 - f_{3x}g_3}{f_3 g_3}, \quad v = -a^2\frac{\gamma_1}{\gamma_2}\frac{g_3 h_3}{f_3^2} + a\frac{a\gamma_1 - 2\gamma_3}{2\gamma_2}\left(\frac{f_{3x}^2}{f_3^2} - \frac{g_{3x}^2}{g_3^2} - \frac{f_{3xx}}{f_3} + \frac{g_{3xx}}{g_3}\right) \tag{45}$$

with

$$g_3 = \frac{\alpha}{\sqrt{2}}\mathrm{e}^{\alpha^2 t}\Big[1 + 2(\mathrm{e}^{\xi_1+2\theta_1} + \mathrm{e}^{\xi_2+2\theta_2} + \mathrm{e}^{\xi_3+2\theta_3}) + 4\mathrm{e}^{\xi_1+\xi_2+2\theta_1+2\theta_2+A_{12}} + 4\mathrm{e}^{\xi_1+\xi_3+2\theta_1+2\theta_3+A_{13}} + \\ + 4\mathrm{e}^{\xi_2+\xi_3+2\theta_2+2\theta_3+A_{23}} + 8\mathrm{e}^{\xi_1+\xi_2+\xi_3+2\theta_1+2\theta_2+2\theta_3+A_{12}+A_{13}+A_{23}}\Big] \tag{46}$$

$$h_3 = \frac{\alpha}{\sqrt{2}}\mathrm{e}^{-\alpha^2 t}\Big[1 + 2(\mathrm{e}^{\xi_1-2\theta_1} + \mathrm{e}^{\xi_2-2\theta_2} + \mathrm{e}^{\xi_3-2\theta_3}) + 4\mathrm{e}^{\xi_1+\xi_2-2\theta_1-2\theta_2+A_{12}} + 4\mathrm{e}^{\xi_1+\xi_3-2\theta_1-2\theta_3+A_{13}} + \\ + 4\mathrm{e}^{\xi_2+\xi_3-2\theta_2-2\theta_3+A_{23}} + 8\mathrm{e}^{\xi_1+\xi_2+\xi_3-2\theta_1-2\theta_2-2\theta_3+A_{12}+A_{13}+A_{23}}\Big] \tag{47}$$

$$f_3 = 1 + 2(\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \mathrm{e}^{\xi_3}) + 4\mathrm{e}^{\xi_1+\xi_2+A_{12}} + 4\mathrm{e}^{\xi_1+\xi_3+A_{13}} + 4\mathrm{e}^{\xi_2+\xi_3+A_{23}} + 8\mathrm{e}^{\xi_1+\xi_2+\xi_3+A_{12}+A_{13}+A_{23}} \tag{48}$$

If selecting:

$$f^{(1)} = 2(\mathrm{e}^{\xi_1} + \mathrm{e}^{\xi_2} + \cdots + \mathrm{e}^{\xi_n}), \quad g^{(1)} = \sqrt{2}\alpha(\mathrm{e}^{\xi_1+2\theta_1} + \mathrm{e}^{\xi_2+2\theta_2} + \cdots + \mathrm{e}^{\xi_n+2\theta_n}) \tag{49}$$

$$h^{(1)} = \sqrt{2}\alpha(\mathrm{e}^{\xi_1-2\theta_1} + \mathrm{e}^{\xi_2-2\theta_2} + \cdots + \mathrm{e}^{\xi_n-2\theta_n}) \tag{50}$$

by induction we can finally reach the n-soliton solutions (12) determined by eqs. (13)-(16) of eqs. (1) and (2). Thus, we finish the proof of Theorem 3.

Conclusion

In summary, we have bilinearized the WBK eqs. (1) and (2) and obtained new one-soliton solutions (36), two-soliton solutions (44), three-soliton solutions (45) and the uniform formulae of n-soliton solutions (12) through Hirota's bilinear method. In the procedure of extending Hirota's bilinear method to eqs. (1) and (2), one of the key steps is taking the transformations (5) to reduce eqs. (1) and (2) to the AKNS eq. (6) which provide with convenience for the bilinear forms (9) and (10) of eqs. (1) and (2). Recently, fractional-order differential calculus and its applications have attached much attention [26-29]. How to construct multi-soliton solutions of non-linear PDE with fractional derivatives is worthy of study.

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Nomenclature

a, b, c – constants, [–]
D_t, D_x – Hirota's differential operators, [–]
e – base of natural logarithms, [–]
i, j – natural numbers, [–]
k_j – constant, [–]
l, n – natural numbers, [–]
t – time, [s]
x – displacement, [m]

Greek symbols

α, β, γ – constants, [–]
γ_i – constant, [–]
θ_j, θ_l – constants, [–]
μ_j, μ_l – integers, [–]
ξ_j^0 – constant, [–
ω_j – constant, [–]

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