SOLUTIONS OF THE HEAT-CONDUCTION MODEL DESCRIBED BY FRACTIONAL EMDEN-FOWLER TYPE EQUATION

by

Chunfu WEI* and Huanhuan WANG
School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo, China

Original scientific paper
https://doi.org/10.2298/TSCI17S1113W

In this paper, we presented a reliable algorithm to solve the singularity initial value problems of the time-dependent fractional Emden-Fowler type equations by homotopy analysis method. The approximate solutions of the problems are obtained.

Key words: *fractional Emden-Fowler type equations, homotopy analysis method, approximate solution, Caputo fractional derivative*

Introduction

In this paper, we consider the following singularity initial value problem for the time-dependent fractional Emden-Fowler type equation:

$$\mathrm{D}_x^{2\alpha} y + \frac{r}{x^\alpha}\mathrm{D}_x^{\alpha} y + af(x,t)g(y) = \mathrm{D}_t^{\alpha} y, \quad 0 < \alpha \le 1 \tag{1}$$

with the initial value conditions

$$y(0,t) = A, \quad y'_x(0,t) = 0 \tag{2}$$

where A, a, and r are constants, $f(x,t)$ and $g(y)$ are the given functions, and $\mathrm{D}^\alpha(\alpha > 0)$ is the fractional derivative in Caputo sense, see, for example, [1-6].

When $\alpha = 1$, eq. (1) was adopted to describe the heat-conduction model of the human head [7]. The singularity behavior that occurs at the origin is the main difficulty in the analysis of eqs. (1) and (2).

Recently, several researchers discussed such initial value problems [7-11]. Various numerical methods, such as, were applied to this kind of the initial value problems, see [12-20] and the cited references therein. The purpose of the present work is to use the homotopy analysis method (HAM) [7, 12] to obtain the approximate analytical solutions of eq. (1).

Basic definitions of fractional derivatives

In this section, we give some basic definitions and properties of fractional derivatives which will be used in this paper [1-6, 19-23].

Definition 1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\mu$, $n \in N$.

The Riemann-Liouville fractional integral operator is defined.

* Corresponding author, e-mail: mathwcf@163.com

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f(x)$ in the power-law kernel is defined as [1, 19]:

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s) \mathrm{d}s$$

$$J^0 f(x) = f(x)$$

Properties of the operator J^{α} can be found in and we mention only the following: for $\alpha, \beta \ge 0$, $x > 0$ and $\gamma > -1$:

$$J^{\alpha} J^{\beta} f(x) = J^{\alpha+\beta} f(x)$$

$$J^{\alpha} J^{\beta} f(x) = J^{\beta} J^{\alpha} f(x)$$

$$J^{\alpha}(x^{\gamma}) = \frac{\Gamma(\gamma+1)}{\Gamma(1+\alpha+\gamma)} x^{\gamma+\alpha}$$

Definition 3. The Caputo fractional derivative of $f(x)$ in the power-law kernel is defined [1, 19]:

$$\mathrm{D}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s$$

where $m-1 < \alpha \le m$, $m \in N^{+}$, $x > 0$ and $f \in C_{-1}^{m}$.

We recall here two of its basic properties [19]:

$$\mathrm{D}^{\alpha} J^{\alpha} f(x) = f(x)$$

$$J^{\alpha} \mathrm{D}^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{x^k}{k!}, \quad x > 0$$

Definition 4. The Riemann-Liouville-Yang fractional integral operator of order $1 > \alpha > 0$ of a function $f(x)$ in the power-law kernel is defined [4]:

$${}_{0}^{\mathrm{RLT}}J_{\tau}^{(\alpha)} f(\tau) = \frac{1}{\Gamma(-\alpha)} \int_0^{\tau} \frac{f(t)}{(\tau-t)^{1+\alpha}} \mathrm{d}t$$

Definition 5. The Riemann-Liouville-Yang type fractional derivative of $f(x)$ of order $1 > \alpha > 0$ in the power-law kernel is defined [4]:

$${}_{0}^{\mathrm{RLT}}\mathrm{D}_{t}^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\theta)^{\alpha} f(\theta) \mathrm{d}\theta$$

Definition 6. The Liouville-Caputo-Yang type fractional derivative of $f(x)$ of order $1 > \alpha > 0$ in the power-law kernel is defined [4]:

$${}_{0}^{\mathrm{CT}}\mathrm{D}_{t}^{(\alpha)} f(t) = \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\theta)^{\alpha} f^{(1)}(\theta) \mathrm{d}\theta$$

We notice that the Caputo fractional and Liouville-Caputo-Yang derivatives are different because their kernels are singular and non-singular, respectively.

Definition 7. The Yang-Machado-Dumitru fractional integral operator of order $1>\alpha>0$ of a function $f(x)$ in the power-law kernel is defined [3]:

$$ {}^{\mathrm{RLT}}_{0}J_{\tau}^{(\alpha)} f(\tau) = f(\tau) + \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{f(t)}{(\tau - t)^{1-\alpha}} \mathrm{d}t $$

Definition 8. The Yang-Machado-Dumitru type fractional derivative of $f(x)$ of order $1>\alpha>0$ in the power-law kernel of first kind is defined [3]:

$$ {}^{\mathrm{RLT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{t} E_{\alpha}\left[-(t-\theta)^{\alpha}\right] f(\theta)\mathrm{d}\theta $$

where the Mittag-Leffler function is defined:

$$ E_{\alpha}\left(t^{\alpha}\right) = \sum_{\eta=0}^{\infty} \frac{t^{\eta\alpha}}{\Gamma(\eta\alpha + 1)} $$

Definition 9. The Yang-Machado-Dumitru type fractional derivative of $f(x)$ of order $1>\alpha>0$ in the power-law kernel of first kind is defined [3]:

$$ {}^{\mathrm{CT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = \int_0^{t} E_{\alpha}\left[-(t-\theta)^{\alpha}\right] f^{(1)}(\theta)\mathrm{d}\theta $$

The relationship between the Yang-Machado-Dumitru fractional derivatives of first kind is [3]:

$$ {}^{\mathrm{CT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = {}^{\mathrm{RLT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) - E_{\alpha}(-t^{\alpha}) f(0) $$

Definition 10. The Yang-Machado-Dumitru fractional integral operator of order $1>\alpha>0$ of a function $f(x)$ in the power-law kernel of second kind is defined [3]:

$$ {}^{\mathrm{RLT}}_{0}J_{\tau}^{(\alpha)} f(\tau) = f(\tau) - \frac{1}{\Gamma(\alpha)} \int_0^{\tau} \frac{f(t)}{(\tau - t)^{1-\alpha}} \mathrm{d}t $$

Definition 11. The Yang-Machado-Dumitru type fractional derivative of $f(x)$ of order $1>\alpha>0$ in the power-law kernel of second kind is defined [3, 20]:

$$ {}^{\mathrm{RLT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^{t} E_{\alpha}\left[(t-\theta)^{\alpha}\right] f(\theta)\mathrm{d}\theta $$

Definition 12. The Yang-Machado-Dumitru type fractional derivative of $f(x)$ of order $1>\alpha>0$ in the power-law kernel of second kind is defined [3, 20]:

$$ {}^{\mathrm{CT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = \int_0^{t} E_{\alpha}\left[(t-\theta)^{\alpha}\right] f^{(1)}(\theta)\mathrm{d}\theta $$

The relationship between the Yang-Machado-Dumitru fractional derivatives of second kind can be written [3]:

$$ {}^{\mathrm{CT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) = {}^{\mathrm{RLT}}_{0}\mathrm{D}_{t}^{(\alpha)} f(t) - E_{\alpha}(t^{\alpha}) f(0) $$

For more definitions of fractional derivatives in the different kernels, see [20-24].

Homotopy analysis method

Here we follow the paper [12]. For the convenience of readers, we still give the basic idea of the method as follows.

Let us consider the following differential equation:

$$N[u(r,t)] = 0 \tag{3}$$

where N is a linear or non-linear operator, r and t are independent variables, and $u(r, t)$ is an unknown function. For simplicity, here we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Yang *et al.* [3] constructs the so called zero order deformation equation:

$$(1-p)L\left[\varphi(r,t;p) - u_0(x,t)\right] = phH(r,t)N\left[\varphi(r,t;p)\right] \tag{4}$$

where $p \in [0,1]$ is the embedding parameter, h – the non-zero auxiliary parameter, $H(r,t)$ – the non-zero auxiliary function, L – the auxiliary linear operator, $u_0(r,t)$ – the initial guess of $u(r,t)$, and $\varphi(r,t;p)$ – the unknown function. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds:

$$\varphi(r,t;0) = u_0(r,t),\ \varphi(r,t;1) = u(r,t)$$

respectively. Thus, as p increases from 0 to 1, the solution $\varphi(r,t;p)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\varphi(r,t;p)$ in Taylor series with respect to p, we have:

$$\varphi(r,t;p) = u_0(r,t) + \sum_{k=1}^{+\infty} u_k(r,t)p^k \tag{5}$$

where

$$u_k(r,t) = \frac{1}{k!}\frac{\partial^k \varphi(r,t;p)}{\partial p^k}\bigg|_{p=0}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (5) converges at $p = 1$, then we have:

$$u(r,t) = u_0(r,t) + \sum_{k=1}^{+\infty} u_k(r,t) \tag{6}$$

Define the vector:

$$\vec{u}_n = \{u_0(r,t),\ u_1(r,t),\ \ldots,\ u_n(r,t)\}$$

Differentiating eq. (4) k times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $k!$, we obtain the k^{th} order deformation equation:

$$L\left[u_k(r,t) - \chi_k u_{k-1}(x,t)\right] = hH(r,t)R_k(\vec{u}_k, r, t) \tag{7}$$

where

$$R_k(\vec{u}_{k-1}, r, t) = \frac{1}{(k-1)!}\frac{\partial^{k-1} N\left[\varphi(r,t;p)\right]}{\partial p^{k-1}}\bigg|_{p=0}$$

and

$$\chi_k = \begin{cases} 0, & k \le 1, \\ 1, & k > 1 \end{cases}$$

Applying L^{-1} on both side of eq. (7), we get:

$$u_k(r,t) = u_{k-1} + hL^{-1}\left[H(r,t)R_k(\vec{u}_k, r, t)\right] \tag{8}$$

In this way, it is easily to obtain u_k for $k \geq 1$, at M^{th} order, we have:

$$u(r,t) = \sum_{k=0}^{M} u_k(r,t)$$

When $M \to \infty$, we get an accurate approximation of the original eq. (3).

The solutions of eq. (1)

To solve the problem of eqs. (1) and (2) by HAM, we choose the initial approximation:

$$y_0(x) = A$$

and the linear operator:

$$L\left[\phi(x,t;p)\right] = x^{\alpha}\mathrm{D}_x^{2\alpha}\phi(x,t;p) + r\mathrm{D}_x^{\alpha}\phi(x,t;p) \tag{9}$$

Furthermore, we construct the 0^{th}-order deformation equation:

$$\begin{gathered}(1-p)L\left[\varphi(x,t;p) - u_0(x,t;0)\right] = \\ = ph\left[x\mathrm{D}_x^{2\alpha}\varphi + r\mathrm{D}_x^{\alpha}\varphi + ax^{\alpha}f(x,t)g(y) - \mathrm{D}_t^{\alpha}\varphi\right]\end{gathered} \tag{10}$$

Obviously, when $p = 0$ and $p = 1$, we have:

$$\varphi(x,t;0) = y_0(x,t), \quad \varphi(x,t;1) = y(x,t)$$

Expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have:

$$\varphi(x,t;p) = y_0(x,t) + \sum_{k=1}^{+\infty} y_k(x,t)p^k \tag{11}$$

where $y_k(x,t)$ $(k = 1, 2, \cdots)$ will be determined later.

Note that the previous series contains the convergence control parameter h. If it is chosen so property that the previous series is convergent at $p = 1$, then we obtain:

$$y(x,t) = \sum_{k=0}^{\infty} y_k(x,t) \tag{12}$$

Substituting eq. (11) into the 0^{th}-order deformation eq. (11), equating the coefficients of the like powers of p, we have k^{th} deformation equation:

$$L\left[y_k(x,t) - \chi_k y_{k-1}(x,t)\right] = hR_k(\vec{y}_k), \ \ k \geq 1 \tag{13}$$

subject to initial condition:

$$y_k(0,t) = 0, \ \ y'_{kx}(0,t) = 0$$

where

$$R_k(\vec{y}_{k-1}) = x^{\alpha}\mathrm{D}_x^{2\alpha}y_{k-1} + r\mathrm{D}_x^{\alpha}y_{k-1} + ax^{\alpha}f(x,t)G_{k-1} - x^{\alpha}\mathrm{D}_t^{\alpha}y_{k-1}$$

and

$$G_{k-1} = \frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial p^{k-1}}\left[g\left(\sum_{i=0}^{\infty} y_i p^i\right)\right]_{p=0}$$

The solution of the k^{th} deformation eq. (13) for $k \geq 1$ can be obtained:

$$y_k(x,t) = \chi_k y_{k-1}(x,t) + hL^{-1}\left[R_k(\vec{y}_k)\right]$$

Finally, we have:

$$y(x,y) = y_0(x,t) + y_1(x,t) + y_2(x,t) + \cdots$$

As an example, if we let $A = 1$, and $f(x,t) = x^m t^s$, $g(y) = y^n$, then we give:

$$y_0(x) = 1$$

$$y_1(x,t) = \frac{ah\Gamma(m+1)\Gamma(m+1+\alpha)t^s x^{m+2\alpha}}{\Gamma(m+1+2\alpha)[\Gamma(m+1+\alpha)+r\Gamma(m+1)]}$$

$$y_2(x,t) = \frac{ah(1+h)\Gamma(1+m)\Gamma(m+1+\alpha)t^s x^{m+2\alpha}}{\Gamma(m+1+2\alpha)[\Gamma(m+1+\alpha)+r\Gamma(m+1)]} +$$

$$+\frac{\Gamma(m+1)\Gamma(m+1+\alpha)\Gamma(2m+2\alpha+1)\Gamma(2m+3\alpha+1)h^2a^2nt^{2s}x^{2m+4\alpha}}{\Gamma(m+1+2\alpha)\Gamma(2m+4\alpha+1)[\Gamma(m+1+\alpha)+r\Gamma(m+1)][\Gamma(2m+3\alpha+1)+r\Gamma(2m+2\alpha+1)]}+$$

$$+\frac{-\Gamma(m+1)\Gamma^2(m+1+\alpha)\Gamma(2m+2\alpha+1)\Gamma(m+2\alpha+1)\Gamma(s+1)h^2a\ nt^{s-\alpha}x^{m+4\alpha}}{\left\{\begin{matrix}\Gamma(m+1+2\alpha)\Gamma(m+3\alpha+1)[\Gamma(m+1+\alpha)+r\Gamma(m+1)][\Gamma(m+2\alpha+1)+r\Gamma(m+\alpha+1)]\cdot \\ \cdot\Gamma(s+1-\alpha)\end{matrix}\right\}}$$

Preceding in this manner the rest of the components y_k, $k \geq 3$ can be completely obtained, and the series solutions are thus entirely determined.

When $f(x,t) = x^m$, $A = 1$, $r = 1$, and $\alpha = 1$, by the previous solutions, taking $h = -1$, we can obtain the following stationary solution:

$$y_0(x) = 1$$

$$y_1(x) = -\frac{a}{(m+2)(m+3)}x^{m+2}$$

$$y_2(x) = \frac{a^2 n}{2(m+2)^2(m+3)(2m=5)}x^{2m+4}$$

which is agreement with the result obtained in [1].

Conclusion

By using the HAM, we presented a reliable algorithm to solve singularity initial value problems of time-dependent fractional Emden-Fowler type equations. The method is accurate and efficient.

Acknowledgment

This paper was partially supported by science and technology project of Henan Province (Grant No. 152102210317), the Key Research Project for the University of Henan Province (Grant No. 15A440008), the Fundamental Research Funds for the Universities of Henan

Province (Grant No. NSFRF140138), and the Doctoral Foundation at Henan Polytechnic University in China (Grant No. B2015-52).

Nomenclature

D^α – fractional derivative, [–]
r – space co-ordinate, [m]
t – time co-ordinate, [s]

Greek symbol

α – order, [Pa]

References

[1] Yang, X. J., *et al.*, *Local Fractional Integral Transforms and their Applications*, Academic Press, New York, USA, 2005
[2] Yang, X. J., General Fractional Derivatives: A Tutorial Comment, *Proceedings*, Symposium on Advanced Computational Methods for Linear and Nonlinear Heat and Fluid Flow 2017 & Advanced Computational Methods in Applied Science 2017& Fractional (Fractal) Calculus and Applied Analysis 2017, Songjiang, Shanghai, China
[3] Yang, X. J., *et al*., Anomalous Diffusion Models with General Fractional Derivatives within the Kernels of the Extended Mittag-Leffler Type Functions, *Romanian Reports in Physics*, *69* (2017), 4, 115
[4] Yang, X. J., New Rheological Problems Involving General Fractional Derivatives within Nonsingular Power-Law Kernel, *Proceedings of the Romanian Academy - Series A*, *69* (2017), 3, in press
[5] Yang, X. J., *et al.*, A New Fractional Operator of Variable Order: Application in the Description of Anomalous Diffusion, *Physica A: Statistical Mechanics and its Applications*, *481* (2017), Sept., pp. 276-283
[6] Yang, X. J., Fractional Derivatives of Constant and Variable Orders Applied to Anomalous Relaxation Models in Heat-Transfer Problems, *Thermal Science*, *21* (2017), 3, pp. 1161-1171
[7] Wang, H. H., Hu, Y., Solutions of Fractional Emden-Fowler Equations by Homotopy Analysis Method, *Journal of Advances in Mathematics*, *13* (2017), 1, pp. 7042-7047
[8] Chowdhury, M. S. H., Hashim, I., Solutions of Emden-Fowler Equations by Homotopy Perturbation Method, *Nonlinear Analysis Real World Applications*, *10* (2009), 1, pp. 104-115
[9] Wazwaz, A. M., A New Algorithm for Solving Differential Equations of Lane-Emden Type, *Applied Mathematics & Computation, 118* (2001), 2-3, pp. 287-310
[10] Wong, J. S. W., On the Generalized Emden–Fowler Equation, *Siam Review, 17* (1975), 2, pp. 339-360
[11] Shang, X, *et al.*, An Efficient Method for Solving Emden-Fowler Equations, *Journal of the Franklin Institute,* 346 (2009), 2, pp. 889-897
[12] Liao, S. J., Homotopy Analysis Method: a New Analytical Technique for Nonlinear Problems, *Communications in Nonlinear Science and Numerical Simulation, 2* (1997), 2, pp. 95-100
[13] Baleanu, D., *et al.*, An Existence Result for a Superlinear Fractional Differential Equation, *Applied Mathematics Letters, 23* (2010), 9, pp. 1129-1132
[14] Dehghan, M., Fatemeh S., Solution of an Integro-Differential Equation Arising in Oscillating Magnetic Fields Using He's Homotopy Perturbation Method, *Progress in Electromagnetics Research*, *78* (2008), 1, pp. 361-376
[15] Khan, J. A., *et al*., Numerical Treatment of Nonlinear Emden-Fowler Equation Using Stochastic Technique, *Annals of Mathematics and Artificial Intelligence*, *63* (2011), 2, pp. 185-207
[16] Chowdhury, S. H., A Comparison between the Modified Homotopy Perturbation Method and Adomian Decomposition Method for Solving Nonlinear Heat Transfer Equations, *Journal of Applied Sciences, 11* (2011), 7, pp. 1416-1420
[17] Wazwaz, A. M, *et al.*, Solving the Lane-Emden-Fowler Type Equations of Higher Orders by the Adomian Decomposition Method, *Computer Modeling in Engineering & Sciences*, *100* (2014), 6, pp. 507-529
[18] Kaur, H, *et al.*, Haar Wavelet Approximate Solutions for the Generalized Lane-Emden Equations Arising in Astrophysics, *Computer Physics Communications*, *184* (2013), 9, pp. 2169-2177
[19] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000, pp. 1021-1032
[20] Yang, X. J., New General Fractional-Order Rheological Models within Kernels of Mittag-Leffler Functions, *Romanian Reports in Physics*, *69* (2017), 4, 118
[21] Yang, X. J., *et al.*, Some New Applications for Heat and Fluid Flows Via Fractional Derivatives without Singular Kernel, *Thermal Science*, *20* (2016), Suppl. 3, pp. S833-S839

[22] Gao, F., *et al.*, Fractional Maxwell Fluid with Fractional Derivative without Singular Kernel, *Thermal Science*, *20* (2016), Suppl. 3, pp. S871-S877
[23] Yang, X. J., *et al*., A New Fractional Derivative without Singular Kernel: Application to the Modelling of the Steady Heat Flow, *Thermal Science*, *20* (2016), 2, pp. 753-756
[24] Yang, A. M., *et al.*, On Steady Heat Flow Problem Involving Yang-Srivastava-Machado Fractional Derivative without Singular Kernel, *Thermal Science*, *20* (2016), Suppl. 3, pp. S717-S721

Paper submitted: March 10, 2017
Paper revised: May 1, 2017
Paper accepted: June 28, 2017