

AN INTEGRAL TRANSFORM APPLIED TO SOLVE THE STEADY HEAT TRANSFER PROBLEM IN THE HALF-PLANE

by

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An integral transform operator

$$\mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \Pi(t) e^{-i\lambda t} dt$$

is considered to solve the steady heat transfer problem in this paper. The analytic technique is illustrated to be applicable in the solution of a 1-D Laplace equation in the half-plane. The results are interesting as well as potentially useful in the linear heat transfer problems.

Key words: heat transfer, Laplace equation, analytic solution, integral transform

Introduction

The PDE were used to describe the heat transfer problems, such as diffusion [1-4], and Laplace equation [5-7]. The PDE in the steady heat transfer problems [5-9] are a second-order PDE named after Pierre-Simon Laplace who first researched its properties. Its extended versions were discussed by different researcher in the sense of the fractional- and fractal-order space operators. For example, the fractional Laplacian equations with the critical Sobolev exponent were considered in [10]. The weak and numerical solutions were reported in [11, 12]. The local fractional Laplace equation with local fractional derivative was reported in [13, 14].

The (linear) Laplace equations in the different conditions were solved by the integral transforms [15]. A novel Fourier-like integral transform operator was proposed to find the exact solutions for a steady heat transfer problem [16]. The Fourier-like integral transform technique for the Laplace equation has not yet considered. The brief target of the manuscript is to consider the proposed method for find the analytic solution for the Laplace equation.

A novel integral transform technique

In this paper, we present the properties of the novel integral transform operator which was proposed in [16] and used in the paper.

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The integral transform of the function $\Pi(t)$ is [16]:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \Pi(t) e^{-i\lambda t} dt \quad (1)$$

where \mathfrak{A} is the Fourier-like integral transform operator.

The inverse Fourier-like integral transform operator is defined [16]:

$$\Pi(t) = \mathfrak{A}^{-1}[\Pi(\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Pi(\lambda) \lambda e^{i\lambda t} d\lambda \quad (2)$$

provided the integral exists for some λ .

As a direct result of eqs. (21) and (22), an integral formula is presented [16]:

$$\Pi(t) = \mathfrak{A}^{-1}[\Pi(\lambda)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lambda \left[\frac{1}{\lambda} \int_{-\infty}^{\infty} \Pi(t) e^{-i\lambda t} dt \right] e^{i\lambda t} d\lambda \quad (3)$$

The properties of the Fourier-like integral transform operator are [16].

(U1) Suppose that $\Pi(\omega) = \mathbb{F}[\Pi(t)]$ and $\Pi(\kappa) = \mathfrak{A}[\Pi(t)]$. Then we have:

$$\Pi(\omega) = \lambda \Pi(\lambda) \quad \text{and} \quad \Pi(\lambda) = \frac{1}{\omega} \Pi(\omega) \quad (4)$$

(U2) If $\Pi(t) = e^{-at} u(t)$, then we have:

$$\Pi(\lambda) = \frac{1}{\lambda(a + i\lambda)} \quad (5)$$

(U3) If $\Pi(t) = \delta(t)$, then we have:

$$\Pi(\omega) = \frac{1}{\lambda} \quad (6)$$

(U4) Let $\Pi(\lambda) = \mathfrak{A}[\Pi(t)]$. Then:

$$\mathfrak{A}[\Pi(t - a)] = e^{-i\kappa a} \Pi(\kappa) \quad (7)$$

(U5) Let $\Pi(\lambda) = \mathfrak{A}[\Pi(t)]$. Then:

$$\mathfrak{A}\left[\frac{d\Pi(t)}{dt}\right] = i\lambda \Pi(\lambda) \quad (8)$$

As the direct results, we have:

$$\mathfrak{A}\left[\frac{d^2\Pi(t)}{dt^2}\right] = -\lambda^2 \Pi(\lambda) \quad (9)$$

$$\mathfrak{A}\left[\frac{d^3\Pi(t)}{dt^3}\right] = -i\lambda^3 \Pi(\lambda) \quad (10)$$

$$\mathfrak{A}\left[\frac{d^4\Pi(t)}{dt^4}\right] = \lambda^4 \Pi(\lambda) \quad (11)$$

$$\mathfrak{A}\left[\frac{d^n \Pi(t)}{dt^n}\right] = (i\lambda)^n \Pi(\lambda) \quad (12)$$

(U6) Let $\Pi(\lambda) = \mathfrak{A}[\Pi(t)]$ and $\Theta(\lambda) = \mathfrak{A}[\Theta(t)]$. Then:

$$\mathfrak{A}[\Pi(t) + \Theta(t)] = \Pi(\lambda) + \Theta(\lambda) \quad (13)$$

(U7) Let $\Pi(\lambda) = \mathfrak{A}[\Pi(t)]$ and $\Theta(\lambda) = \mathfrak{A}[\Theta(t)]$. Then:

$$\mathfrak{A}\left[\int_{-\infty}^{\infty} \Pi(t-\tau)\Theta(\tau) d\tau\right] = \lambda \Pi(\lambda)\Theta(\lambda) \quad (14)$$

(U7) Let $\Pi(\lambda) = \mathfrak{A}[\Pi(t)]$. Then:

$$\mathfrak{A}\left[\int_{-\infty}^{\infty} \Pi(t-\nu)\Theta(\nu) d\nu\right] = \lambda \Pi(\lambda)\Theta(\lambda) \quad (15)$$

The properties of the Fourier-like integral transform operator are presented.

(S1) If $\Pi(t) = t^n e^{iat}$, then we have:

$$\Pi(\lambda) = \frac{2\pi i^n}{\lambda} \delta^{(n)}(\lambda - a) \quad (16)$$

where $\delta(\lambda)$ is the Dirac function [15].

Proof. According to the definition of the Fourier-like integral transform operator, we have:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} t^n e^{-iat} e^{-i\lambda t} dt = \frac{2\pi i^n}{\lambda} \delta^{(n)}(\lambda - a)$$

where the formula [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-iatx} e^{-i\lambda t} dt = \sqrt{2\pi} i^n \delta^{(n)}(\lambda - a)$$

is valid.

(S2) If $\Pi(t) = \text{sgn}(t)$, where $\text{sgn}(t)$ is the sign function [15], then we have:

$$\Pi(\lambda) = \frac{2}{i\lambda^2} \quad (17)$$

Proof. Using the definition of the Fourier-like integral transform operator, we obtain:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \text{sgn}(t) e^{-i\lambda t} dt = \frac{2}{i\lambda^2}$$

where there is [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \text{sgn}(t) e^{-i\lambda t} dt = \sqrt{\frac{2}{\pi}} \frac{1}{i\lambda}$$

(S3) If $\Pi(t) = 1/(t^2 + a^2)$, where $a > 0$, then we have:

$$\Pi(\lambda) = \frac{\pi e^{-a|\lambda|}}{a\lambda} \quad (18)$$

Proof. Considering the following relationship [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{t^2 + a^2} e^{-i\lambda t} dt = \sqrt{\frac{\pi}{2}} \frac{e^{-a|\lambda|}}{a}$$

we have:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \frac{1}{t^2 + a^2} e^{-i\lambda t} dt = \frac{\pi e^{-a|\lambda|}}{a\lambda}$$

(S4) If $\Pi(t) = e^{-at^2}$, where $a > 0$, then we have:

$$\Pi(\lambda) = \frac{1}{\lambda} e^{-a\frac{\lambda^2}{4a}} \quad (19)$$

Proof. Using the following relationship [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} e^{-a\frac{\lambda^2}{4a}}$$

we get:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{-at^2} e^{-i\lambda t} dt = \frac{1}{\lambda} e^{-a\frac{\lambda^2}{4a}}$$

(S5) If $\Pi(t) = e^{-a|t|}$, where $a > 0$, then we have:

$$\Pi(\lambda) = \frac{\pi a}{\lambda(a^2 + \lambda^2)} \quad (20)$$

Proof. With use of the relationship [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\lambda t} dt = \sqrt{\frac{\pi}{2}} \frac{a}{a^2 + \lambda^2}$$

we have:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{-a|t|} e^{-i\lambda t} dt = \frac{\pi a}{\lambda(a^2 + \lambda^2)}$$

(S6) If $\Pi(t) = e^{iat}$, where $a > 0$, then we have:

$$\Pi(\lambda) = \frac{2\pi\delta(\lambda - a)}{\lambda} \quad (21)$$

Proof. We have [15]:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iat} e^{-i\lambda t} dt = \sqrt{2\pi}\delta(\lambda - a)$$

such that:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} e^{iat} e^{-i\lambda t} dt = \frac{2\pi\delta(\lambda - a)}{\lambda}$$

(S7) If $\Pi(t) = \sin t$, then we have:

$$\Pi(\lambda) = \frac{\pi[\delta(\lambda - 1) - \delta(\lambda + 1)]}{\lambda i} \quad (22)$$

Proof. We have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin t e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{it} - e^{-it}}{2i} e^{-i\lambda t} dt = \frac{1}{2i} [\sqrt{2\pi}\delta(\lambda - 1) - \sqrt{2\pi}\delta(\lambda + 1)]$$

such that:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \sin t e^{-i\lambda t} dt = \frac{\pi[\delta(\lambda - 1) - \delta(\lambda + 1)]}{\lambda i}$$

(S8) If $\Pi(t) = \cos t$, then we have:

$$\Pi(\lambda) = \frac{\pi[\delta(\lambda - 1) + \delta(\lambda + 1)]}{\lambda} \quad (23)$$

Proof. We have:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos t e^{-i\lambda t} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{it} + e^{-it}}{2} e^{-i\lambda t} dt = \frac{1}{2} [\sqrt{2\pi}\delta(\lambda - 1) + \sqrt{2\pi}\delta(\lambda + 1)]$$

such that:

$$\Pi(\lambda) = \mathfrak{A}[\Pi(t)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \cos t e^{-i\lambda t} dt = \frac{\pi[\delta(\lambda - 1) + \delta(\lambda + 1)]}{\lambda}$$

Solving the 1-D Laplace equation

In this section, we handle a Dirichlet's problem for a 1-D Laplace equation in the half-plane.

Let us consider a 1-D Laplace equation in the half-plane [15]:

$$\frac{\partial^2 \Theta(x, y)}{\partial x^2} + \frac{\partial^2 \Theta(x, y)}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y \geq 0 \quad (24)$$

with the boundary conditions:

$$\Theta(x, 0) = g(x), \quad -\infty < x < \infty \quad (25)$$

$$\Theta(x, y) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty, \quad y \rightarrow \infty \quad (26)$$

Introducing the Fourier-like integral transform operator with respect to x , we have:

$$\Theta(\lambda, y) = \mathfrak{A}[\Theta(x, y)] = \frac{1}{\lambda} \int_{-\infty}^{\infty} \Theta(x, y) e^{-i\lambda x} dx \quad (27)$$

which leads to:

$$\frac{\partial^2 \Theta(\lambda, y)}{\partial y^2} - \lambda^2 \Theta(\lambda, y) = 0 \quad (28)$$

where

$$\Theta(\lambda, 0) = g(\lambda) \quad (29)$$

and

$$\Theta(\lambda, y) \rightarrow 0 \quad \text{as } y \rightarrow \infty \quad (30)$$

Thus, finding the solution of eq. (27), we obtain from eqs. (28) and (29) that:

$$\Theta(\lambda, y) = g(\lambda) e^{-|\lambda|y} \quad (31)$$

Thus, we have:

$$\mathfrak{A} \left[\sqrt{\frac{2}{\pi}} \frac{y}{x^2 + y^2} \right] = \frac{1}{\lambda} e^{-|\lambda|y} \quad (32)$$

such that:

$$\Theta(\lambda, y) = \lambda g(\lambda) \left(\frac{1}{\lambda} e^{-|\lambda|y} \right) \quad (33)$$

Thus, we have from eq. (33):

$$\Theta(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\nu) d\nu}{(x-\nu)^2 + y^2}, \quad y \geq 0 \quad (34)$$

which is in agreement with the result from Fourier transform [15].

Conclusion

In the present work, a novel Fourier-like integral transform technique was used to solve the PDE in heat transfer. The analytical solution of the Laplace equation was obtained. Comparing with the result from Fourier transform, the presented method is efficient, accurate and alternative for finding the linear PDE from the problems in science and engineering.

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Nomenclature

t – temperature, [K]

x, y – space co-ordinates, [m]

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