A DIFFERENTIAL-INTEGRAL TRANSFORM METHOD FOR SOLVING THE 1-D HEAT DIFFUSION EQUATION

by

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In this paper, we address a new computational method, which is called the differential-integral transform method, to handle the 1-D diffusion equation.

Key words: *analytic solution, heat diffusion equation, differential-integral transform method*

Introduction

The 1-D heat diffusion equation [1, 2] was considered to describe the physical behavior of the heat flow in the rod. To show the physical behaviors of the heat flows, there are many techniques for find the analytical, approximate, numerical, and exact solutions, such as the finite volume method [3], the mesh-free method [4], the blow-up of finite difference method [5], the immersed interface method [6], the quasi-Newton iterative method [7], the fast Cartesian method [8], integral transforms [2, 9], the coupling method [10], and others [11].

As the expanded versions of the heat diffusion equation, the fractional heat diffusion equations involving the fractional derivative of the constant and variable orders were adopted to develop the anomalous behaviors in the heat transfer phenomena [12-16]. Meanwhile, the fractal heat diffusion equations defined on fractal sets were discussed in [17-20] based on the local fractional derivative and integral operators [21, 22].

The differential transform method (see [23]) was developed to solve the linear and non-linear PDE in mathematical physics [24-26]. For example, the linear and non-linear Klein-Gordon and heat transfer equations were solved in [26, 27]. A Sumudu-like integral transform method was proposed to solve the heat transfer problem [28]. A coupling technique for the differential and Sumudu-like integral transform methods have not yet developed. In this paper, we consider the 1-D heat diffusion equation [21]:

$$\frac{\partial N(x,t)}{\partial t} = \lambda \frac{\partial^2 N(x,t)}{\partial x^2} \tag{1}$$

where λ is the diffusion constant, and $N(x,t)$ is the temperature.

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The main aim of the manuscript is to derive the method to solve the 1-D heat diffusion eq. (1).

The method proposed

In this section, we present the differential transform methods and the differential-integral transform method used in the paper.

The 1-D differential transform method

The 1-D differential transform of the function $\Lambda(x)$ is defined as [23]:

$$\Lambda(\kappa)=\frac{1}{\kappa!}\left[\frac{\partial^{\kappa}\Lambda(x)}{\partial x^{\kappa}}\right] \tag{2}$$

where $\Lambda(x)$ is the original function, and $\Lambda(\kappa)$ is the transformed function.

The 1-D differential inverse transform of the function is defined [23]:

$$\Lambda(x)=\sum_{\kappa=0}^{\infty}x^{\kappa}\Lambda(\kappa) \tag{3}$$

Table 1. The properties of the 1-D differential transform

The 1-D differential transforms	The 1-D differential inverse transforms
$\Lambda(x)=\Lambda_1(x)+\Lambda_2(x)$	$\Lambda(\kappa)=\Lambda_1(\kappa)+\Lambda_2(\kappa)$
$\Lambda(x)=\tau\Lambda_1(x)$	$\Lambda(\kappa)=\tau\Lambda_1(\kappa)$, τ is a constant
$\Lambda(x)=\Lambda_1(x)\Lambda_2(x)$	$\Lambda(\kappa)=\sum_{l=0}^{\kappa}\Lambda_1(l)\Lambda_2(\kappa-l)$
$\Lambda(x)=\mathrm{e}^{x}$	$\Lambda(\kappa)=\frac{1}{\kappa!}$
$\Lambda(x)=\frac{\mathrm{d}^n\Lambda_1(x)}{\mathrm{d}x^n}$	$\Lambda(\kappa)=\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa+n)$

The properties of the 1-D differential transform [23] are listed in tab. 1.

The 2-D differential transform method

The 2-D differential transform of the function $\Lambda(x,t)$ is defined [24-28]:

$$\Lambda(\kappa,\sigma)=\frac{1}{\kappa!\sigma!}\left[\frac{\partial^{\kappa+\sigma}\Lambda(x,t)}{\partial x^{\kappa}\partial t^{\sigma}}\right] \tag{4}$$

where $\Lambda(x,t)$ is the original function, and $\Lambda(\kappa,\sigma)$ is the transformed function.

The differential inverse transform of the function is defined [24-28]:

$$\Lambda(x,t)=\sum_{\kappa=0}^{\infty}\sum_{\sigma=0}^{\infty}x^{\kappa}t^{\sigma}\Lambda(\kappa,\sigma) \tag{5}$$

The prosperities of the 2-D differential transform method [24-28] are listed in tab. 2.

The Sumudu-like integral transform and its properties

The Sumudu-like integral transform of $\Lambda(t)$ is defined [2, 29]:

$$\Lambda(\varpi)=\Upsilon\left[\Lambda(t)\right]=\int_0^{\infty}\Lambda(t)\mathrm{e}^{-\frac{t}{\varpi}}\mathrm{d}t,\quad t>0 \tag{6}$$

and its inverse Sumudu-like integral transform is defined as [2,29]:

$$\Lambda(t)=\Upsilon^{-1}\left[\Lambda(\varpi)\right] \tag{7}$$

Table 2. The properties of the 2-D differential transform

The 2-D differential transforms	The 2-D differential inverse transforms
$\Lambda(x,t)=\Lambda_1(x,t)+\Lambda_2(x,t)$	$\Lambda(\kappa,\sigma)=\Lambda_1(\kappa,\sigma)+\Lambda_2(\kappa,\sigma)$
$\Lambda(x,t)=\tau\Lambda_1(x,t)$	$\Lambda(\kappa,\sigma)=\tau\Lambda_1(\kappa,\sigma)$, τ is a constant.
$\Lambda(x,t)=\Lambda_1(x,t)\Lambda_2(x,t)$	$\Lambda(\kappa,\sigma)=\sum_{l=0}^{\kappa}\sum_{m=0}^{\sigma}\Lambda_1(l,\sigma-m)\Lambda_2(\kappa-l,\sigma)$
$\Lambda(x,t)=\dfrac{\partial^n\Lambda_1(x,t)}{\partial x^n}$	$\Lambda(\kappa,\sigma)=\dfrac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa+n,\sigma)$

The properties of the Sumudu-like integral transform [2, 29] are listed in tab. 3.

Table 3. The properties of the Sumudu-like integral transform

Functions	The Laplace-like integral transforms
$\Pi(t)=\Lambda_1(t)\pm\pm\Lambda_2(t)$	$Y[\Lambda_1(t)\pm\Lambda_2(t)]=\Lambda_1(\varpi)\pm\pm\Lambda_2(\varpi)$
$\Pi(t)=1$	$Y[1]=\varpi$
$\Pi(t)=e^{\tau t}$	$Y[e^{\tau t}]=\dfrac{\varpi}{1-\tau\varpi}$
$\Pi(t)=\Lambda_1(ct)$	$Y[\Lambda(ct)]=\dfrac{1}{c}\Lambda\left(\dfrac{\varpi}{c}\right)$
$\Pi(t)=\Lambda^{(1)}(t)$	$Y\left[\Lambda^{(1)}(t)\right]=\dfrac{1}{\varpi}\Lambda(\varpi)-\Lambda(0)$

The differential-integral transform method

The differential-integral transform method of the function $\Lambda(x,t)$ is defined:

$$\Lambda(\kappa,\varpi)=Y\left[\Lambda(\kappa,t)\right]=\frac{1}{\kappa!}\left[\frac{\partial^\kappa\Lambda(x,\varpi)}{\partial x^\kappa}\right] \tag{8}$$

where $\Lambda(x,t)$ is the original function, and $\Lambda(\kappa,\varpi)$ is the transformed function.

The inverse differential-integral transform method of the function is defined:

$$\Lambda(x,t)=Y^{-1}\left[\sum_{\kappa=0}^{\infty}x^\kappa\Lambda(\kappa,\varpi)\right] \tag{9}$$

The properties of the differential-integral transform method are given:

(C1) If $\Lambda(x,t)=\Lambda_1(x,t)+\Lambda_2(x,t)$, then we have $\Lambda(\kappa,\varpi)=\Lambda_1(\kappa,\varpi)+\Lambda_2(\kappa,\varpi)$.

Proof. According to the definition of the differential-integral transform method, we have:

$$\Lambda(\kappa,\varpi)=Y\left[\Lambda(x,t)\right]=Y\left\{\frac{\partial^\kappa\left[\Lambda_1(x,t)+\Lambda_2(x,t)\right]}{\partial x^\kappa}\right\}=$$
$$=\frac{1}{\kappa!}\left\{\frac{\partial^\kappa\left[\Lambda_1(x,\varpi)+\Lambda_2(x,\varpi)\right]}{\partial x^\kappa}\right\}=\Lambda_1(\kappa,\varpi)+\Lambda_2(\kappa,\varpi) \tag{10}$$

(C2) If $\Lambda(x,t)=\tau\Lambda_1(x,t)$, then we have $\Lambda(\kappa,\varpi)=\tau\Lambda_1(\kappa,\varpi)$.

Proof. By the definition of the differential-integral transform method, we have:

$$\Lambda(\kappa,\varpi)=Y\left[\Lambda(x,t)\right]=Y\left\{\frac{\partial^\kappa\left[\tau\Lambda_1(x,t)\right]}{\partial x^\kappa}\right\}=\frac{1}{\kappa!}\left\{\frac{\partial^\kappa\left[\tau\Lambda_1(x,\varpi)\right]}{\partial x^\kappa}\right\}=\tau\Lambda_1(\kappa,\varpi) \tag{11}$$

(C3) If $\Lambda(x,t)=\mathrm{e}^x$, then we have $\Lambda(\kappa,\varpi)=\varpi/\kappa!$.

Proof. Applying the definition of the differential-integral transform method, we have:

$$\Lambda(\kappa,\varpi)=\Upsilon\left[\Lambda(x,t)\right]=\Upsilon\left[\frac{\partial^\kappa(\mathrm{e}^x)}{\partial x^\kappa}\right]=\frac{\varpi}{\kappa!} \tag{12}$$

(C4) If $\Lambda(x,t)=\mathrm{e}^t$, then we have $\Lambda(\kappa,\varpi)=\varpi/(1-\tau\varpi)$.

Proof. Following the definition of the differential-integral transform method, we obtain:

$$\Lambda(\kappa,\varpi)=\Upsilon\left[\mathrm{e}^t\right]=\frac{\varpi}{1-\tau\varpi} \tag{13}$$

(C5) If $\Lambda(x,t)=\mathrm{d}^n\Lambda_1(x,t)/\mathrm{d}x^n$, then we have:

$$\Lambda(\kappa,\varpi)=\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa+n,\varpi)$$

Proof. By using the definition of the differential-integral transform method, we obtain:

$$\Lambda(\kappa,\varpi)=\Upsilon\left[\Lambda(x,t)\right]=\Upsilon\left[\frac{\partial^\kappa\Lambda(x,t)}{\partial x^\kappa}\right]=\Upsilon\left[\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa+n,t)\right]=\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa+n,\varpi) \tag{14}$$

(C6) If $\Lambda(x,t)=[\mathrm{d}\Lambda_1(x,t)]/\mathrm{d}t$, then we have:

$$\Lambda(\kappa,\varpi)=\frac{1}{\varpi}\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa,\varpi)$$

Proof. By using the definition of the differential-integral transform method, we obtain:

$$\Lambda(\kappa,\varpi)=\Upsilon\left[\Lambda(x,t)\right]=\Upsilon\left[\frac{\partial^\kappa}{\partial x^\kappa}\frac{\mathrm{d}^n\Lambda_1(x,t)}{\mathrm{d}t^n}\right]=$$

$$=\Upsilon\left[\frac{1}{\varpi}\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa,\varpi)\right]=\frac{1}{\varpi}\frac{(\kappa+n)!}{\kappa!}\Lambda_1(\kappa,\varpi) \tag{15}$$

Solving the 1-D diffusion equation

In this section, the differential-integral transform method is used to solve the 1-D diffusion equation with the initial-boundary value conditions.

Let us consider the 1-D heat diffusion eq. (1) with initial-boundary value conditions:

$$N(0,t)=\mathrm{e}^t \tag{16}$$

$$\frac{\partial}{\partial x}N(0,t)=\mathrm{e}^t \tag{17}$$

$$N(x,0)=\mathrm{e}^x \tag{18}$$

Taking the Sumudu-like integral transform of eq. (1) with respective to t, we obtain:

$$\frac{1}{\varpi}N(x,\varpi)-\mathrm{e}^x=\lambda\frac{\partial^2N(x,\varpi)}{\partial x^2} \tag{19}$$

From eq. (19) we have:

$$\Lambda\left(x,\varpi\right)=\sum_{\kappa=0}^{\infty}x^{\kappa}\Lambda\left(x,\varpi\right) \tag{20}$$

Such that:

$$\frac{1}{\varpi}N\left(\kappa,\varpi\right)-\frac{1}{\kappa!}=\lambda\frac{\left(\kappa+2\right)!}{\kappa!}\Lambda_1\left(\kappa+2,\varpi\right) \tag{21}$$

where

$$\frac{\partial}{\partial x}N\left(0,\varpi\right)=\frac{\varpi}{1-\varpi} \tag{22}$$

$$N\left(0,\varpi\right)=\frac{\varpi}{1-\varpi} \tag{23}$$

The components of $N(\kappa,\varpi)$ are listed in tab. 4. Therefore, we have:

$$N\left(x,t\right)=\Upsilon^{-1}\left[\frac{\varpi}{1-\varpi}\sum_{\kappa=0}^{3}\frac{x^{\kappa}}{3!}\right]=\mathrm{e}^{t}\sum_{\kappa=0}^{3}\frac{x^{\kappa}}{\kappa!} \tag{24}$$

which leads to:

$$N\left(x,t\right)=\mathrm{e}^{t}\sum_{\kappa=0}^{\infty}\frac{x^{\kappa}}{\kappa!}=\mathrm{e}^{t}\mathrm{e}^{x}=\mathrm{e}^{x+t} \tag{25}$$

We notice that eqs. (24) and (25) are the numerical and exact solution of eq. (1), respectively.

Table 4. The components of $N(\kappa,\varpi)$

$N(0,\varpi)$	$\frac{\varpi}{1-\varpi}$
$N(1,\varpi)$	$\frac{\varpi}{1-\varpi}\frac{1}{1!}$
$N(2,\varpi)$	$\frac{\varpi}{1-\varpi}\frac{1}{2!}$
$N(3,\varpi)$	$\frac{\varpi}{1-\varpi}\frac{1}{3!}$

Conclusion

In this work, the differential-integral transform method was presented for the first time. The numerical solution for the 1-D diffusion equation with the initial-boundary value conditions was obtained. The proposed method is efficient and accurate.

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Nomenclature

$N(x,t)$ – temperature, [K]
t – time co-ordinate, [s]
x – space co-ordinate, [m]

Greek symbols

κ – diffusion coefficient, [m^2s^{-1}]

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