ANALYTICAL SOLUTIONS OF LINEAR DIFFUSION AND WAVE EQUATIONS IN SEMI-INFINITE DOMAINS BY USING A NEW INTEGRAL TRANSFORM

by

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Recently, a new integral transform similar to Sumudu transform has been proposed by Yang [1]. Some of the properties of the integral transform are expanded in the present article. Meanwhile, new applications to the linear wave and diffusion equations in semi-infinite domains are discussed in detail. The proposed method provides an alternative approach to solve the partial differential equations in mathematical physics.

Key words: *integral transform, diffusion equation, wave equation, analytical solution*

Introduction

Integral transforms (IT) have been the best-known methods to solve the differential and integral equations in recent years [1-8]. Currently, the conventional IT of two types, *e. g.*, Laplace transform (LT) [1, 4] and Sumudu transform (ST) [1, 3], were widely used in solving the PDE arising from mathematical physics.

Recently, a new IT similar to the ST has been proposed in [1]. Some properties of the new IT were analyzed and the differential equation (DE) in the steady heat transfer (SHT) problem was solved based on the new IT. However, the properties of the new IT have not been comprehensively obtained and the new IT is only just used to solve the ODE. Thus, in this paper we first consummate the properties of the new IT and apply the new IT to solve three PDE, for example, linear diffusion equation (DE) and linear wave equation (WE) in semi-infinite domains.

A new IT

A new IT of the function $l(t)$, denoted by $L(\theta)$, is given [1]:

$$L(\theta)=R\left[l(t)\right]=\int_0^\infty l(t)\mathrm{e}^{-\frac{t}{\theta}}\mathrm{d}t, \quad t>0 \tag{1}$$

provided that the IT operator exists for some θ, where R is the IT operator.

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The properties adopted in this paper are presented as follows.

(T1) If $L(\theta) = R[l(t)]$ and $G(\theta) = R[g(t)]$, then we have [1]:

$$R\left[al(t) + bg(t)\right] = aL(\theta) + bG(\theta) \tag{2}$$

where a and b are constants.

(T2) Let $R[l(t)] = L(\theta)$. Then we have:

$$R\left[l^{(2)}(t)\right] = \frac{1}{\theta^2}L(\theta) - \frac{1}{\theta}l(0) - l^{(1)}(0) \tag{3}$$

Proof.

$$R\left[l^{(2)}(t)\right] = \int_0^{\infty} l^{(2)}(t)\mathrm{e}^{\frac{-t}{\theta}}\mathrm{d}t = \frac{1}{\theta}\left[\frac{1}{\theta}L(\theta) - l(0)\right] - l^{(1)}(0) = \frac{1}{\theta^2}L(\theta) - \frac{1}{\theta}l(0) - l^{(1)}(0) \tag{4}$$

Generally,

$$R\left[l^{(n)}(t)\right] = \frac{1}{\theta^n}L(\theta) - \frac{1}{\theta^{n-1}}l(0) - \frac{1}{\theta^{n-2}}l^{(1)}(0) - \cdots - l^{(n-1)}(0) \tag{5}$$

where $l^{(n)}(t)$ is the derivative of l of order n with respect to t.

For $n = 1$, eq. (5) was discussed [1].

(T3) Let $R[l_1(t)] = L_1(\theta)$ and $R[l_2(t)] = L_2(\theta)$. Then we have:

$$R\left[l_1(t)l_2(t)\right] = L_1(\theta)L_2(\theta) \tag{6}$$

where the convolution of $l_1(t)$ and $l_2(t)$ is defined as [3]:

$$l_1(t) * l_2(t) = \int_0^t l_1(\tau)l_2(t-\tau)\mathrm{d}\tau \tag{7}$$

Proof.

$$R\left[l_1(t) * l_2(t)\right] = \int_0^{+\infty}\left[\int_0^t l_1(\tau)l_2(t-\tau)\mathrm{d}\tau\right]\mathrm{e}^{-\frac{t}{\theta}}\mathrm{d}t \tag{8}$$

By exchanging integral order, we have:

$$R\left[l_1(t) * l_2(t)\right] = \int_0^{+\infty} l_1(\tau)\left[\int_\tau^{+\infty} l_2(t-\tau)\mathrm{e}^{-\frac{t}{\theta}}\mathrm{d}t\right]\mathrm{d}\tau \tag{9}$$

Taking $t - \tau = u$, we have:

$$\int_\tau^{+\infty} l_2(t-\tau)\mathrm{e}^{-\frac{t}{\theta}}\mathrm{d}t = \int_0^{+\infty} l_2(u)\mathrm{e}^{-\frac{u+\tau}{\theta}}\mathrm{d}u = \mathrm{e}^{-\frac{\tau}{\theta}}\int_0^{+\infty} l_2(u)\mathrm{e}^{-\frac{u}{\theta}}\mathrm{d}u = \mathrm{e}^{-\frac{\tau}{\theta}}L_2(\theta) \tag{10}$$

Thus, we have from eq. (10) that:

$$R\left[l_1(t) * l_2(t)\right] = \int_0^{+\infty} l_1(\tau)\mathrm{e}^{-\frac{\tau}{\theta}}L_2(\theta)\mathrm{d}\tau = L_2(\theta)\int_0^{+\infty} l_1(\tau)\mathrm{e}^{-\frac{\tau}{\theta}}\mathrm{d}\tau = L_1(\theta)L_2(\theta) \tag{11}$$

(T4) Suppose $R[l(t)] = L(\theta)$ and $l(t) = c$, where c is a constant. Then we have:

$$R(c) = c\theta \tag{12}$$

Proof.

$$R(c) = \int_0^\infty c\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t = c\int_0^\infty \mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t = c\theta \tag{13}$$

(T5)

$$R(t^n) = (\theta)^{n+1} n! \tag{14}$$

Proof.

$$R(t^n) = \int_0^\infty t^n \mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t = -\theta\int_0^\infty t^n \mathrm{d}e^{\frac{-t}{\theta}} = n\theta\int_0^\infty t^{n-1}\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t = (\theta)^{n+1} n! \tag{15}$$

(T6) If $R[l_1(t)] = L_1(\theta)$ and $\lim\limits_{\theta\to 0}[(1/\theta)L(\theta)]$ exists, then we have:

$$l(0) = \lim_{\theta\to 0}\frac{1}{\theta}L(\theta) \tag{16}$$

Proof.
For $n = 1$, taking the limit of $\theta \to 0$ on both sides of eq. (5), we have:

$$\lim_{\theta\to 0} R\left[l'(t)\right] = \lim_{\theta\to 0}\left[\frac{1}{\theta}L(\theta) - l(0)\right] = \lim_{\theta\to 0}\frac{1}{\theta}L(\theta) - l(0) \tag{17}$$

The term in the left of eq. (17) is:

$$\lim_{\theta\to 0} R\left[l'(t)\right] = \lim_{\theta\to 0}\int_0^{+\infty} l'(t)\mathrm{e}^{-\frac{t}{\theta}}\,\mathrm{d}t = \int_0^{+\infty}\lim_{\theta\to 0} l'(t)\,\mathrm{e}^{-\frac{t}{\theta}}\,\mathrm{d}t = 0 \tag{18}$$

Thus, we obtain from eq. (18) that:

$$l(0) = \lim_{\theta\to 0}\frac{1}{\theta}L(\theta) \tag{19}$$

(T7) Let $R[l(t)] = L(\theta)$. Then we have:

$$R\left[tl(t)\right] = \theta^2 L^{(1)}(\theta) \tag{20}$$

where $L^{(1)}$ is the derivative of L of order one.
Proof.

$$L^{(1)}(\theta) = \left[\int_0^\infty l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t\right]^{(1)} = \frac{1}{\theta^2}\int_0^\infty tl(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t = \frac{1}{\theta^2}R\left[tl(t)\right] \tag{21}$$

(T8) Suppose that $l(t)$ is a function of the period of T and let $R[l(t)] = L(\theta)$. Then we have:

$$R\left[l(t)\right] = \frac{1}{1-\mathrm{e}^{-\frac{T}{\theta}}}\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t \tag{22}$$

Proof.

$$R\left[l(t)\right]=\int_0^\infty l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t=\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t+\int_T^\infty l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t \tag{23}$$

Taking $u=t-T$, we have:

$$R\left[l(t)\right]=\int_0^\infty l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t=\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t+\int_0^\infty l(u)\,\mathrm{e}^{\frac{-u}{\theta}}\,\mathrm{e}^{-\frac{T}{\theta}}\,\mathrm{d}u=$$

$$=\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t+\mathrm{e}^{-\frac{T}{\theta}}\int_0^\infty l(u)\,\mathrm{e}^{\frac{-u}{\theta}}\,\mathrm{d}u=\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t+\mathrm{e}^{-\frac{T}{\theta}}R\left[l(t)\right] \tag{24}$$

Thus,

$$R\left[l(t)\right]=\frac{1}{1-\mathrm{e}^{-\frac{T}{\theta}}}\int_0^T l(t)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t \tag{25}$$

(T9)

$$R\left[H(t-t_0)\right]=\theta\mathrm{e}^{\frac{-t_0}{\theta}} \tag{26}$$

where the unit step function is defined by [2]:

$$H(t-t_0)=\begin{cases}0, & t\le t_0\\ 1, & t>t_0\end{cases} \tag{27}$$

Proof.

$$R\left[H(t-t_0)\right]=\int_0^\infty H(t-t_0)\,\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t=\int_{t_0}^\infty \mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t=\theta\mathrm{e}^{\frac{-t_0}{\theta}} \tag{28}$$

(T10)

$$R\left[\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right]=\theta\mathrm{e}^{-\frac{a}{\sqrt{\theta}}} \tag{29}$$

where the complementary error function is defined as, see, for example, [2]:

$$\mathrm{erfc}\left(\frac{a}{2\sqrt{t}}\right)=1-\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}}\mathrm{e}^{-x^2}\,\mathrm{d}x \tag{30}$$

Proof.

$$R\left(1-\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}}\mathrm{e}^{-x^2}\,\mathrm{d}x\right)=\theta-\int_0^\infty\left[\frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}}\mathrm{e}^{-x^2}\,\mathrm{d}x\right]\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t \tag{31}$$

By interchanging the order of integration for eq. (31) we can obtain:

$$\frac{2}{\sqrt{\pi}}\int_0^\infty\mathrm{e}^{-x^2}\,\mathrm{d}x\int_0^{a^2/4x^2}\mathrm{e}^{\frac{-t}{\theta}}\,\mathrm{d}t=\frac{2}{\sqrt{\pi}}\int_0^\infty\mathrm{e}^{-x^2}\theta\left[1-\exp\left(-\frac{a^2}{4x^2\theta}\right)\right]\mathrm{d}x=$$

$$=\theta\frac{2}{\sqrt{\pi}}\left\{\int_0^\infty e^{-x^2}dx-\int_0^\infty\left[\exp-\left(x^2+\frac{a^2}{4x^2\theta}\right)\right]dx\right\} \quad (32)$$

Thus, we have from eq. (32) that:

$$\int_0^\infty\exp\left[-\left(x^2+\frac{a^2}{4\theta x^2}\right)\right]dx=\frac{1}{2}\int_0^\infty\left(1-\frac{a}{2\sqrt{\theta}x^2}\right)\exp\left[-\left(x+\frac{a}{2\sqrt{\theta}x}\right)^2+\frac{a}{\sqrt{\theta}}\right]dx+$$
$$+\int_0^\infty\left(1+\frac{a}{2\sqrt{\theta}x^2}\right)\exp\left[-\left(x-\frac{a}{2\sqrt{\theta}x}\right)^2-\frac{a}{\sqrt{\theta}}\right]dx=\frac{1}{2}e^{-\frac{a}{\sqrt{\theta}}}\int_{-\infty}^{+\infty}e^{-y^2}dy=\frac{\sqrt{\pi}}{2}e^{-\frac{a}{\sqrt{\theta}}} \quad (33)$$

Consequently,

$$R\left[\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right]=\theta-\theta\frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}e^{-\frac{a}{\sqrt{\theta}}}\right)=\theta e^{-\frac{a}{\sqrt{\theta}}} \quad (34)$$

(T11) If $l(t)=e^{at}$, then we have [1]:

$$R(e^{at})=\frac{\theta}{1-a\theta} \quad (35)$$

where a is a constant.

Solving a class of the PDE in semi-infinite domains

At first, we consider the linear DE in the semi-infinite domain [2, 9]:

$$\frac{\partial\vartheta_\sigma(x,t)}{\partial t}-k\frac{\partial^2\vartheta_\sigma(x,t)}{\partial x^2}=0,\quad x>0,\quad t>0 \quad (36)$$

where k is the diffusivity.

The initial-boundary value conditions (IBVC) are:

$$\vartheta_\sigma(x,0)=0,\quad \vartheta_\sigma(0,t)=b,\quad |\vartheta_\sigma(x,t)|\le M,\quad M>0 \quad (37)$$

where b is a constant.

By taking the new IT of eq. (36) and the IBVC with respect to t we present in the form:

$$\frac{d^2}{dx^2}\hat{\vartheta}_\sigma(x,\varpi)-\frac{1}{k\varpi}\hat{\vartheta}_\sigma(x,\varpi)=0 \quad (38)$$

subject to the initial value:

$$\hat{\vartheta}_\sigma(0,\varpi)=b\varpi \quad (39)$$

In view of eq. (38), we obtain its general solution in the following form:

$$\hat{\vartheta}_\sigma(x,\varpi)=C_1\exp\left(\sqrt{\frac{1}{k\varpi}}x\right)+C_2\exp\left(-\sqrt{\frac{1}{k\varpi}}x\right) \quad (40)$$

where C_1 and C_2 are the constants to be determined.

Since $\vartheta_\sigma(x,t)$ is bounded, we easily yield:

$$C_1 = 0 \tag{41}$$

Evidently, eq. (40) becomes:

$$\hat{\vartheta}_\sigma(x,\varpi) = C_2 \exp\left(-\sqrt{\frac{1}{k\varpi}}x\right) \tag{42}$$

It follows from eq. (39) that:

$$C_2 = b\varpi \tag{43}$$

Substituting eq. (43) into eq. (42), we obtain:

$$\hat{\vartheta}_\sigma(x,\varpi) = b\varpi \exp\left(-\sqrt{\frac{1}{k\varpi}}x\right) \tag{44}$$

Application of the inverse transform of the new IT for eq. (44) gives:

$$R^{-1}\left[b\varpi \exp\left(-\sqrt{\frac{1}{k\varpi}}x\right)\right] = b\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) \tag{45}$$

Thus, the analytical solution of eq. (36) in the semi-infinite domain is:

$$\vartheta_\sigma(x,t) = b\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) \tag{46}$$

Equation (46) is in line with the result by the LT, see, *e. g.*, [2].

As the second example, let us consider the linear WE in the semi-infinite domain [10]:

$$\frac{\partial^2 \vartheta_\upsilon(x,t)}{\partial t^2} - a^2 \frac{\partial^2 \vartheta_\upsilon(x,t)}{\partial x^2} = g, \quad x > 0, \ t > 0 \tag{47}$$

where a is the wave speed, and g is a constant.

The IBVC are presented:

$$\vartheta_\upsilon(x,0) = 0, \quad \frac{\partial \vartheta_\upsilon(x,0)}{\partial t} = 0, \quad \vartheta_\upsilon(0,t) = 0, \quad \left|\vartheta_\upsilon(x,t) \le M, \quad M > 0\right| \tag{48}$$

Taking the new IT of eq. (47) with respect to *t*, we have the following:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\hat{\vartheta}_\upsilon(x,\varpi) - \frac{1}{a^2}\frac{1}{\varpi^2}\hat{\vartheta}_\upsilon(x,\varpi) = -\frac{\varpi g}{a^2} \tag{49}$$

subject to the initial value:

$$\hat{\vartheta}_\upsilon(0,\varpi) = 0 \tag{50}$$

In view of eq. (49), the analytic formula of the WE in the semi-infinite domain can be expressed:

$$\hat{\vartheta}_\upsilon(x,\varpi) = c_1 \mathrm{e}^{\frac{1}{a\varpi}x} + c_2 \mathrm{e}^{-\frac{1}{a\varpi}x} + \varpi^3 g \tag{51}$$

where c_1 and c_2 are constants to be determined.

From eq. (48), we get:

$$c_1 = 0 \tag{52}$$

Thus, eq. (51) becomes:

$$\hat{\vartheta}_\upsilon(x,\varpi) = c_2 \mathrm{e}^{-\frac{1}{a\varpi}x} + \varpi^3 g \tag{53}$$

Substituting eq. (50) into eq. (53), we easily obtain:

$$c_2 = -\varpi^3 g \tag{54}$$

Hence, eq. (51) can be expressed:

$$\vartheta_\upsilon(x,\varpi) = \varpi^3 g\left(1 - \mathrm{e}^{-\frac{1}{a\varpi}x}\right) \tag{55}$$

which, by using the inverse IT of eq. (55), gives the analytical solution of the form:

$$\vartheta_\upsilon(x,t) = R^{-1}\left[\hat{\vartheta}_\upsilon(x,\varpi)\right] = \frac{g}{2}t^2 - gt\,H\left(t - \frac{x}{a}\right) \tag{56}$$

Equation (56) is in agreement with the result by the LT, see, for example, [2].

Conclusion

In the present paper, some properties of a new IT were developed. It was used to develop to find the analytical solutions for the linear DE and WE in semi-infinite domains. The results are in agreement with those of ST and LT. The technology is accurate and efficient to solve the ODE and PDE in mathematical physics.

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Nomenclature

a	–	the wave number, [m²s⁻²]
k	–	the diffusivity, [Wm⁻¹K⁻¹]
$R[l(t)]$	–	integral transform, [–]
t	–	time, [s]
x	–	space co-ordinate, [m]

Greek symbols

$\vartheta_\sigma(x,t)$	–	temperature, [K]
$\vartheta_\upsilon(x,t)$	–	wave speed, [ms⁻¹]

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