THE FOURIER-YANG INTEGRAL TRANSFORM FOR SOLVING THE 1-D HEAT DIFFUSION EQUATION

by

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A new Fourier-like integral transform (called the Fourier-Yang integral transform)

$$\mathbb{S}\left[\Lambda(t)\right]=\varepsilon\int_{-\infty}^{\infty}\Lambda(t)\mathrm{e}^{-j\varepsilon t}\mathrm{d}t$$

is considered to find the fundamental solutions of the 1-D heat diffusion equation in the different initial conditions.

Key words: *fundamental solution, heat equation, diffusion equation, Fourier-Yang integral transform, Fourier-like integral transform*

Introduction

The PDE in the heat transfer problems are the important topics for scientists and engineers to explore the heat transport in the solid, liquid and gas [1-4]. The heat diffusion equation is one of the interesting PDE for describe the heat transfer theory [5-7] and the diffusion flow in metamorphic rocks [8, 9]. With the aid of the (non-local and local) fractional calculus, the heat diffusion equation can be generalized to fractional diffusion equations [10-12] and local fractional diffusion equations [13-15].

In order to find the solutions for the heat diffusion equations, many technologies, such as the Laplace-like integral transform [5], finite integral transform [16], homology [17], variational iteration [18], alternating-direction implicit [19], immersed interface [20], and the Laplace-like integral transform [21] methods, were developed.

A new Fourier-like integral transform (called the Fourier-Yang integral transform), proposed by Yang [22], was considered to solve the steady heat transfer problem. More integral transforms for solving the heat transfer problems were considered in [23-25]. The aim of the present manuscript is to present the properties of this integral transform and a new application to find the fundamental solution for a 1-D heat diffusion equation.

The Fourier-Yang integral transform

In this section, we introduce the concepts of Fourier and Fourier-Yang integral transforms, and properties of the Fourier-Yang integral transform.

The Fourier integral transform of the function $\Phi(t)$ is given [23]:

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$$\Phi(\theta)=\wp\left[\Phi(t)\right]=:\int_{-\infty}^{\infty}\Phi(t)\mathrm{e}^{-j\theta t}\mathrm{d}t \tag{1}$$

where $\wp$ is the Fourier integral transform operator.

The inverse Fourier integral transform operator of eq. (4) is written [23]:

$$\Phi(t)=\wp^{-1}\left[\Phi(\theta)\right]=:\frac{1}{2\pi j}\int_{-\infty}^{\infty}\Phi(t)\mathrm{e}^{j\theta t}\mathrm{d}\theta \tag{2}$$

where $\wp^{-1}$ is the inverse Fourier integral transform operator.

The Fourier integral formula is given [23]:

$$\Phi(t)=\wp^{-1}\left[\Phi(\theta)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\Phi(t)\mathrm{e}^{-j\theta t}\mathrm{d}t\right]\mathrm{e}^{j\theta t}\mathrm{d}\theta \tag{3}$$

The new Fourier-Yang integral transform of the function $\Lambda(t)$ is given [22]:

$$\Lambda(\varepsilon)=\mathbb{S}\left[\Lambda(t)\right]=:\varepsilon\int_{-\infty}^{\infty}\Lambda(t)\mathrm{e}^{-j\varepsilon t}\mathrm{d}t \tag{4}$$

where $\mathbb{S}$ is the new Fourier-Yang integral transform operator.

The inverse Fourier-Yang integral transform operator is defined [22]:

$$\Lambda(t)=\mathbb{S}^{-1}\left[\Lambda(\varepsilon)\right]=:\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Lambda(\varepsilon)}{\varepsilon}\mathrm{e}^{j\varepsilon t}\mathrm{d}\varepsilon \tag{5}$$

where $\mathbb{S}^{-1}$ is the inverse Fourier-Yang integral transform operator.

The Fourier-Yang integral formula is given [22]:

$$\Lambda(t)=\mathbb{S}^{-1}\left[\Lambda(\varepsilon)\right]=:\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{\varepsilon}\left[\varepsilon\int_{-\infty}^{\infty}\Lambda(t)\mathrm{e}^{-j\varepsilon t}\mathrm{d}t\right]\mathrm{e}^{j\varepsilon t}\mathrm{d}\varepsilon \tag{6}$$

Taking $\varpi = j\varepsilon$, we obtain the Laplace-Carson integral transform of the function $\Omega(t)$ [24]:

$$\Omega(\gamma)=\Re\left[\Omega(t)\right]=:\gamma\int_{0}^{\infty}\Pi(t)\mathrm{e}^{-\gamma t}\mathrm{d}t \tag{7}$$

where $\Re$ is the Laplace-Carson integral transform operator.

Similarly, the inverse Laplace-Carson integral transform operator is presented [24]:

$$\Omega(t)=\Re^{-1}\left[\Omega(\gamma)\right]=:\frac{1}{2\pi j}\int_{\omega_0-i\infty}^{\omega_0+i\infty}\frac{\Omega(\gamma)}{\gamma}\mathrm{e}^{\gamma t}\mathrm{d}\gamma \tag{8}$$

The properties of the Fourier-Yang integral transform operator are as follows [22].

(T1) If $\Lambda(\theta)=\wp[\Lambda(t)]$ and $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$, then we have:

$$\Lambda(\theta)=\frac{1}{\varepsilon}\Lambda(\varepsilon)\quad\text{and}\quad\Lambda(\theta)=\varepsilon\Lambda(\varepsilon) \tag{9}$$

(T2) If $\Lambda(t)=\mathrm{e}^{-at}\varphi(t)$, where $\varphi(t)$ is the Heaviside unit step function, then we have:

$$\Lambda(\varepsilon)=\frac{\varepsilon}{a+j\varepsilon} \tag{10}$$

where a is a constant.
(T3) If $\Lambda(t)=\delta(t)$, where $\delta(t)$ represents the Dirac function, then we have:

$$\Lambda(\varepsilon)=\varepsilon \tag{11}$$

(T4) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$, then we have:

$$\mathbb{S}\left[\Lambda(t-a)\right]=\mathrm{e}^{-ja\varepsilon}\Lambda(\varepsilon) \tag{12}$$

(T5) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$, then we have:

$$\mathbb{S}\left[\frac{\mathrm{d}\Lambda(t)}{\mathrm{d}t}\right]=j\varepsilon\Lambda(\varepsilon) \tag{13}$$

(T6) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$, then we have:

$$\mathbb{S}\left[\frac{\mathrm{d}^2\Lambda(t)}{\mathrm{d}t^2}\right]=-\varepsilon^2\Lambda(\varepsilon) \tag{14}$$

(T7) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$ and $\Theta(\varepsilon)=\mathbb{S}[\Theta(t)]$, then we have:

$$\mathbb{S}\left[\Lambda(t)+\Theta(t)\right]=\Lambda(\varepsilon)+\Theta(\varepsilon) \tag{15}$$

(T8) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$ and $\Theta(\varepsilon)=\mathbb{S}[\Theta(t)]$, then we have:

$$\mathbb{S}\left[\int_{-\infty}^{\infty}\Lambda(t-\tau)\Theta(\tau)\mathrm{d}\tau\right]=\frac{1}{\varepsilon}\Lambda(\varepsilon)\Theta(\varepsilon) \tag{16}$$

(T9) If $\Lambda(\varepsilon)=\mathbb{S}[\Lambda(t)]$, then we have:

$$S\left[\int_{-\infty}^{\infty}\Lambda(t)\mathrm{d}t\right]=\frac{1}{j\varepsilon}\Lambda(\varepsilon) \tag{17}$$

(T10) If $\Lambda(t)=b\mathrm{e}^{-at^2}$, where $a>0$, then we have:

$$\Lambda(\varepsilon)=\frac{b\varepsilon\pi}{\sqrt{\pi a}}\mathrm{e}^{-\frac{\varepsilon^2}{4a}} \tag{18}$$

Proof. We have, by the definition of the Fourier-Yang integral transform, that:

$$\Lambda(\varepsilon)=\varepsilon\int_{-\infty}^{\infty}b\mathrm{e}^{-at^2}\mathrm{e}^{-j\varepsilon t}\mathrm{d}t=\varepsilon\left\{\int_{-\infty}^{\infty}b\mathrm{e}^{\left[-a\left(t+\frac{j\varepsilon}{\sqrt{2a}}\right)^2-\frac{\varepsilon^2}{4a}\right]}\mathrm{d}t\right\}=\varepsilon\left[b\mathrm{e}^{-\frac{\varepsilon^2}{4a}}\int_{-\infty}^{\infty}\mathrm{e}^{-at^2}\mathrm{d}t\right]=\frac{b\varepsilon\pi}{\sqrt{\pi a}}\mathrm{e}^{-\frac{\varepsilon^2}{4a}} \tag{19}$$

where

$$\int_{-\infty}^{\infty}\mathrm{e}^{-at^2}\mathrm{d}t=\sqrt{\frac{\pi}{a}} \tag{20}$$

(T11) If

$$\Lambda(t)=\begin{cases}\mathrm{M}, & |t|\le\dfrac{1}{2}\\ 0\end{cases} \tag{21}$$

then we have:

$$\Lambda(\varepsilon)=\varepsilon\int_{-1/2}^{1/2}\mathrm{M}\mathrm{e}^{-j\varepsilon t}\mathrm{d}t=2\mathrm{M}\sin\frac{\varepsilon}{2} \tag{22}$$

(T12) If

$$\Lambda(t)=\frac{1}{\sqrt{2\pi}\zeta}\mathrm{e}^{-\frac{t^2}{2\zeta^2}}$$

then we have:

$$\Lambda(\varepsilon)=\varepsilon\mathrm{e}^{-\frac{\zeta^2\varepsilon^2}{2}} \tag{23}$$

Proof. By the definition of the Fourier-Yang integral transform we have:

$$\Lambda(\varepsilon)=\varepsilon\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\zeta}\mathrm{e}^{-\frac{t^2}{2\zeta^2}}\mathrm{e}^{-j\varepsilon t}\mathrm{d}t=\varepsilon\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\zeta}\mathrm{e}^{\left[-\frac{1}{2\zeta^2}(t+j\zeta\varepsilon)^2-\frac{\zeta^2\varepsilon^2}{2}\right]}\mathrm{d}t\right]=$$

$$=\varepsilon\left[\frac{1}{\sqrt{2\pi}\zeta}\mathrm{e}^{-\frac{\zeta^2\varepsilon^2}{2}}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{1}{2\zeta^2}t^2}\mathrm{d}t\right]=\varepsilon\mathrm{e}^{-\frac{\zeta^2\varepsilon^2}{2}} \tag{24}$$

The fundamental solution for the 1-D heat diffusion equation

In this section, we use the Fourier-Yang integral transform to solve a 1-D heat diffusion equation with the different initial conditions.

We now consider the initial value problem for a 1-D heat diffusion equation without source or sinks [23]:

$$\frac{\partial\Lambda(x,t)}{\partial t}=\psi\frac{\partial^2\Lambda(x,t)}{\partial x^2},\quad -\infty<x<\infty,\quad 0<t \tag{25}$$

where ψ is the diffusivity constant with the initial condition:

$$\Lambda(x,0)=g(x),\ -\infty<x<\infty \tag{26}$$

We find the Fourier-Yang integral transform for this problem with respect to the space variable x.

Let us consider the following equations:

$$\mathbb{S}\left[\frac{\partial\Lambda(x,t)}{\partial t}\right]=\varepsilon\int_{-\infty}^{\infty}\frac{\partial\Lambda(x,t)}{\partial t}\mathrm{e}^{-j\varepsilon x}\mathrm{d}x=\frac{\partial\Lambda(\varepsilon,t)}{\partial t} \tag{27}$$

$$\mathbb{S}\left[\frac{\partial^2\Lambda(x,t)}{\partial x^2}\right]=\varepsilon\int_{-\infty}^{\infty}\frac{\partial^2\Lambda(x,t)}{\partial x^2}\mathrm{e}^{-j\varepsilon x}\mathrm{d}x=-\varepsilon^2\Lambda(\varepsilon,t) \tag{28}$$

Substituting eqs. (27) and (28) into eq. (25), we have:

$$\frac{\mathrm{d}\Lambda(\varepsilon,t)}{\mathrm{d}t}+\psi\varepsilon^2\Lambda(\varepsilon)=0\,, \qquad 0<t \tag{29}$$

where

$$\Lambda(\varepsilon,0)=\varepsilon g(\varepsilon) \tag{30}$$

Finding the solution of eq. (27), we have:

$$\Lambda(\varepsilon,t)=\varepsilon g(\varepsilon)\mathrm{e}^{-\psi\varepsilon^2 t} \tag{31}$$

Making use of the inverse Fourier-Yang integral transform, we get:

$$\Lambda(x,t)=\mathbb{S}^{-1}\left[\Lambda(\varepsilon,t)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon g(\varepsilon)\mathrm{e}^{-\psi\varepsilon^2 t}}{\varepsilon}\mathrm{e}^{j\varepsilon x}\mathrm{d}\varepsilon=\frac{1}{2\pi}\int_{-\infty}^{\infty}g(\varepsilon)\mathrm{e}^{j\varepsilon x-\psi\varepsilon^2 t}\mathrm{d}\varepsilon$$

From eq. (16), we have:

$$\mathbb{S}\left[\int_{-\infty}^{\infty}\Lambda(x-\tau,t)\Theta(\tau,t)\mathrm{d}\tau\right]=\frac{1}{\varepsilon}\Lambda(\varepsilon,t)\Theta(\varepsilon,t) \tag{32}$$

which leads to:

$$\mathbb{S}^{-1}\left[\frac{1}{\varepsilon}\Lambda(\varepsilon,t)\Theta(\varepsilon,t)\right]=\int_{-\infty}^{\infty}\Lambda(x-\tau,t)\Theta(\tau,t)\mathrm{d}\tau \tag{33}$$

In view of eq. (33), we have:

$$\Lambda(x,t)=\int_{-\infty}^{\infty}g(x-\tau,t)\Theta(\tau,t)\mathrm{d}\tau \tag{34}$$

where

$$\Theta(\tau,t)=S^{-1}\left[\varepsilon\mathrm{e}^{-\psi\varepsilon^2 t}\right] \tag{35}$$

Thus, from eq. (23), we obtain:

$$\Lambda(x,t)=\frac{1}{\sqrt{4\pi\psi t}}\int_{-\infty}^{\infty}g(\tau)\mathrm{e}^{-\frac{(x-\tau)^2}{4\psi t}}\mathrm{d}\tau \tag{36}$$

This result is with agreement with the solution of the 1-D heat diffusion equation by using Fourier transform [23].

Let $\Lambda(x,0)=g(x)=\delta(x)$ in eq. (26). Then, from eq. (36) we have:

$$\Lambda(\varepsilon,t)=\varepsilon^2\mathrm{e}^{-\psi\varepsilon^2 t} \tag{37}$$

With the use of the inverse Fourier-Yang integral transform, we have:

$$\Lambda\left(x,t\right)=\mathbb{S}^{-1}\left[\Lambda\left(\varepsilon,t\right)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^2\mathrm{e}^{-\psi\varepsilon^2 t}}{\varepsilon}\mathrm{e}^{j\varepsilon x}\mathrm{d}\varepsilon=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon\mathrm{e}^{j\varepsilon x-\psi\varepsilon^2 t}\mathrm{d}\varepsilon \tag{38}$$

Thus, we obtain the solution for the 1-D heat diffusion equation:

$$\Lambda(x,t)=\frac{1}{\sqrt{4\pi\psi t}}\int_{-\infty}^{\infty}\delta(\tau)\mathrm{e}^{-\frac{(x-\tau)^2}{4\psi t}}\mathrm{d}\tau \tag{39}$$

which results in:

$$\Lambda(x,t)=\frac{1}{\sqrt{4\pi\psi t}}\mathrm{e}^{-\frac{x^2}{4\psi t}} \tag{40}$$

This result is in accordance with the solution of the 1-D heat diffusion equation by using Fourier-like transform [5].

Let $\Lambda(x,0)=g(x)=\mathrm{e}^{-x^2}$ in eq. (26). Then, from eq. (36) we have the solution in the Fourier-Yang integral transform:

$$\Lambda(\varepsilon,t)=\varepsilon^2\sqrt{\pi}\mathrm{e}^{-\frac{\varepsilon^2}{4}}\mathrm{e}^{-\psi\varepsilon^2 t} \tag{41}$$

By the inverse Fourier-Yang integral transform, we have:

$$\Lambda\left(x,t\right)=\mathbb{S}^{-1}\left[\Lambda\left(\varepsilon,t\right)\right]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^2\sqrt{\pi}\mathrm{e}^{-\frac{\varepsilon^2}{4}}\mathrm{e}^{-\psi\varepsilon^2 t}}{\varepsilon}\mathrm{e}^{j\varepsilon x}\mathrm{d}\varepsilon=\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon\sqrt{\pi}\mathrm{e}^{-\frac{\varepsilon^2}{4}}\mathrm{e}^{j\varepsilon x-\psi\varepsilon^2 t}\mathrm{d}\varepsilon \tag{42}$$

which leads to:

$$\Lambda\left(x,t\right)=\frac{1}{\sqrt{4\pi\psi t}}\int_{-\infty}^{\infty}\mathrm{e}^{-x^2}\mathrm{e}^{-\frac{(x-\tau)^2}{4\psi t}}\mathrm{d}\tau \tag{43}$$

Conclusion

We present the new application of the Fourier-Yang integral transform to solve the initial value problem for the 1-D heat diffusion equation in this work. The fundamental solutions of this problem with the initial conditions were obtained with the use of the Fourier-Yang integral transform. The approach for solving this problem is efficient and accurate.

Nomenclature

t – time, [s]
x – space co-ordinate, [m]

Greek symbols

$\Lambda(x,t)$ – temperature, [K]
ψ – diffusivity constant, [$\mathrm{Wm^{-1}K^{-1}}$]

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