

## A VARIATIONAL ITERATION METHOD INTEGRAL TRANSFORM TECHNIQUE FOR HANDLING HEAT TRANSFER PROBLEMS

by

**Yuejin ZHOU<sup>a</sup>, Shun PANG<sup>b\*</sup>, Guo CHONG<sup>b</sup>, Xiaojun YANG<sup>a,c</sup>,  
Xiaoding XU<sup>c</sup>, and Xiaotong LI<sup>b</sup>**

<sup>a</sup> State Key Laboratory for Geomechanics and Deep Underground Engineering,  
China University of Mining and Technology, Xuzhou, China

<sup>b</sup> School of Mines, China University of Mining and Technology, Xuzhou, China

<sup>c</sup> School of Mechanics and Civil Engineering, China University of Mining and Technology,  
Xuzhou, China

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*In this paper, we consider the heat transfer equations at the low excess temperature. The variational iteration method integral transform technique is used to find the approximate solutions for the problems. The used method is accurate and efficient.*

**Key words:** *heat transfer equation, approximate solution, low excess temperature, variational iteration method integral transform*

### Introduction

The solutions for the heat equations [1] were found by the different methods, such as the variation of parameters [2], similarity variable [3], general optimization [4], finite difference domain decomposition [5], optimal homotopy asymptotic [6], He's homotopy perturbation [7], variational iteration [8], Adomian's decomposition [9, 10], integral transform [11-13], Cornejo-Perez and Rosu [14] methods, and so on.

The variational iteration method (VIM) was proposed to handle the ODE and PDE in [15, 16]. Recently, a coupling method of the VIM and integral transform [13] (called the VIM integral transform technique) was proposed for finding the solution of the diffusion and heat equations [17]. The technology has not yet been applied to handle the heat transfer equations at the low excess temperature [14]. The aim of the manuscript is to use the VIM integral transform technique to find the analytical approximate solutions for the heat transfer equations at the low excess temperature.

### Analysis of the method applied

In this section, the VIM integral transform technology is presented in [17].

#### *A novel integral transform method*

The integral transform of the function  $\psi(t)$  is defined [13, 17]:

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\* Corresponding author, e-mail: 17798819060@163.com

$$\psi(\omega) = Y[\psi(t)] = \int_0^{\infty} \psi(t) e^{-\frac{t}{\omega}} dt, \quad \tau > 0 \quad (1)$$

where the integral transform operator exists for  $\omega$ .

The inverse integral transform is given by [13, 17]:

$$Y^{-1}\{Y[\psi(t)]\} = Y^{-1}\{\psi(\omega)\} = \psi(t) \quad (2)$$

The properties of the integral transform are briefly presented as follows [13, 17]:

(M1) If  $\psi_1(\omega) = Y[\psi_1(t)]$  and  $\psi_2(\omega) = Y[\psi_2(t)]$ , then we have [13,17]:

$$Y[a\psi_1(t) + b\psi_2(t)] = a\psi_1(\omega) + b\psi_2(\omega) \quad (3)$$

where  $a$  and  $b$  are constants.

(M2) If  $\psi(\omega) = Y[\psi(t)]$ , then we have [13,17]:

$$Y[\psi^{(1)}(t)] = \frac{1}{\omega} \psi(\omega) - \psi(0) \quad (4)$$

(M3) If  $\psi(\omega) = Y[\psi(t)]$ , then we have [13,17]:

$$Y\left[\int_0^t \psi(\tau) d\tau\right] = \omega \psi(\omega) \quad (5)$$

**Table 1. Properties of the integral transform operator**

| Functions                 | The integral transforms                                |
|---------------------------|--|
| $\frac{t^i}{\Gamma(1+i)}$ | $Y\left[\frac{t^i}{\Gamma(1+i)}\right] = \omega^{1+i}$ |
| $\exp(ct)$                | $Y[\exp(ct)] = \frac{\omega}{1-c\omega}$               |

The properties of the integral transform operator used in this paper [13, 17] are listed in tab. 1.

#### The VIM

We now consider the following linear PDE in the operator form:

$$H\varphi + \Sigma\varphi = 0 \quad (6)$$

where  $H = -\partial/\partial t - \beta I$  and  $\Sigma = \alpha \partial^2/\partial x^2$ .

With the help of the idea of the VIM [15, 16], the functional can be written as:

$$\varphi_{n+1}(x, t) = \varphi_n(x, t) + \int_0^t \gamma(t-\tau) [H\varphi_n(x, \tau) + \Sigma\varphi_n(x, \tau)] d\tau \quad (7)$$

Considering the variation of eq. (7) with respect to  $\varphi_n(x, t)$ , we have:

$$\delta\varphi_{n+1}(x, t) = \delta\varphi_n(x, t) + \delta\left[\int_0^t \gamma(t-\tau) [H\varphi_n(x, \tau) + \Sigma\varphi_n(x, \tau)] d\tau\right] = 0 \quad (8)$$

The Lagrange multiplier can be presented [15]:

$$\gamma(t) = -e^{\beta t} \quad (9)$$

Thus, with the aid of eqs. (6) and (9) we can give an iteration algorithm [15], e. g.:

$$\varphi_{n+1}(x, t) = \varphi_n(x, t) - \int_0^t e^{\beta t} [H\varphi_n(x, \tau) + \Sigma\varphi_n(x, \tau)] d\tau \quad (10)$$

Finally, we have:

$$\varphi(x, t) = \lim_{n \rightarrow \infty} \varphi_n(x, t) \quad (11)$$

#### The VIM integral transform method

Following the idea of the VIM integral transform method [17], we can write the functional in the integral transform form, e. g.:

$$\varphi_{n+1}(x, \omega) = \varphi_n(x, \omega) + Y\{\gamma(t)\} Y\{H\varphi_n(x, \tau) + \Sigma\varphi_n(x, \tau)\} \quad (12)$$

Taking the variation of eq. (6) with respect to  $\psi_n(x, \omega)$  into account, we give:

$$\delta\varphi_{n+1}(x, \omega) = \delta\varphi_n(x, \omega) + \delta[Y\{\gamma(t)\} Y\{H\varphi_n(x, t) + \Sigma\varphi_n(x, t)\}] = 0 \quad (13)$$

Thus, we present the integral transform of the Lagrange multiplier given:

$$\gamma(\omega) = -\frac{\omega}{1 - \beta\omega} \quad (14)$$

Thus, from eqs. (12) and (14) we can give an iteration algorithm, namely:

$$\varphi_{n+1}(x, \omega) = \varphi_n(x, \omega) - \frac{\omega}{1 - \beta\omega} Y\{H\varphi_n(x, t) + \Sigma\varphi_n(x, t)\} \quad (15)$$

Therefore, the integral transform solution takes the form:

$$\varphi(x, \omega) = \lim_{n \rightarrow \infty} \varphi_n(x, \omega) \quad (16)$$

Taking inverse integral transform of eq. (16), we obtain the solution to eq. (6), e. g.:

$$\varphi(x, t) = Y^{-1}\left\{\lim_{n \rightarrow \infty} \varphi_n(x, \omega)\right\} \quad (17)$$

#### Solving the heat transfer equation at the low excess temperature

In this section, we present two examples for handling heat transfer equation at the low excess temperature in the different parameters.

##### Example 1

Let us consider the heat transfer equation at the low excess temperature [14]:

$$\frac{\partial \psi(x, t)}{\partial t} = \alpha \frac{\partial^2 \psi(x, t)}{\partial x^2} - \psi(x, t) \quad (18)$$

subject to the initial condition:

$$\psi(x, 0) = x \quad (19)$$

where  $\alpha$  is the thermal diffusivity.

Making use of eq. (15), we have the following iterative algorithm:

$$\psi_{n+1}(x, \omega) = \psi_n(x, \omega) - \frac{\omega}{1 - \omega} Y\left[\frac{\partial \psi_n(x, t)}{\partial t} + \psi_n(x, t) - \alpha \frac{\partial^2 \psi_n(x, t)}{\partial x^2}\right] =$$

$$= \psi_n(x, \omega) - \frac{\omega}{1-\omega} Y \left\{ \frac{\partial \psi_n(x, t)}{\partial t} \right\} - \frac{\omega}{1-\omega} \left[ \psi_n(x, \omega) - \alpha \frac{\partial^2 \psi_n(x, \omega)}{\partial x^2} \right] \quad (20)$$

with the initial condition:

$$\psi_0(x, \omega) = \omega x \quad (21)$$

Taking the first iterative process, we have:

$$\psi_1(x, \omega) = 2\omega x - \frac{\omega}{1-\omega} x \quad (22)$$

which leads to:

$$\psi_1(x, t) = 2x - xe^t \quad (23)$$

The approximate solution of eq. (18) after taking the first iterative process is illustrated in fig. 1.

Similarly, taking the second iterative process, we can obtain:

$$\psi_2(x, \omega) = 2\omega x - \frac{2\omega}{1-\omega} x + \frac{\omega}{1-2\omega} x \quad (24)$$

which reduces to:

$$\psi_2(x, t) = 2x - 2xe^t + xe^{2t} \quad (25)$$

The approximate solution of eq. (18) after taking the second iterative process is displayed in fig. 2.

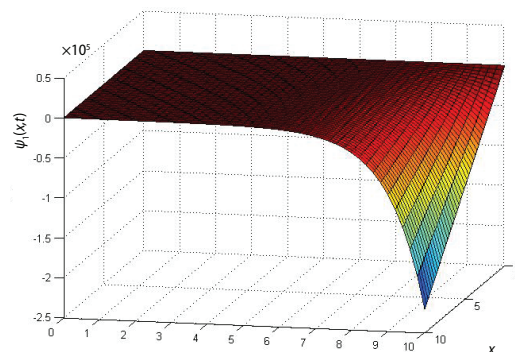


Figure 1. The plot of the approximate solution of eq. (18) after taking the first iterative process (for color image see journal web site)

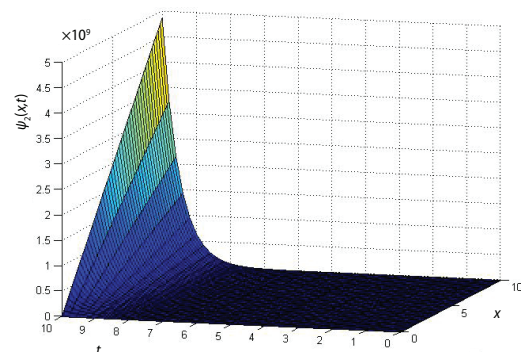


Figure 2. The plot of the approximate solution of eq. (18) after making the second iterative process (for color image see journal web site)

In the similar process, we present the third approximation:

$$\psi_3(x, \omega) = \frac{8\omega}{3} x - \frac{3x\omega}{1-\omega} + \frac{3x\omega}{1-2\omega} e^{2t} - \frac{(2/3)\omega}{1-3\omega} \quad (26)$$

which leads to:

$$\psi_3(x, t) = \frac{8}{3} x - 3xe^t + 3xe^{2t} - \frac{2}{3} xe^{3t} \quad (27)$$

The approximate solution of eq. (18) after taking the third iterative process is demonstrated in fig. 3.

Taking the similar process, we obtain the approximate solution of eq. (18).

### Example 2

We take into account the heat transfer equation at the low excess temperature [14]:

$$\frac{\partial \psi(x, t)}{\partial t} = \alpha \frac{\partial^2 \psi(x, t)}{\partial x^2} - \beta \psi(x, t) \quad (28)$$

with the initial condition:

$$\psi(x, 0) = \exp(x) \quad (29)$$

where  $\alpha$  is the thermal diffusivity and  $\beta$  is a constant.

With the aid of eq. (15), we can structure the iterative algorithm in the form:

$$\begin{aligned} \psi_{n+1}(x, \omega) &= \psi_n(x, \omega) - \frac{\omega}{1 - \beta\omega} Y \left[ \frac{\partial \psi_n(x, t)}{\partial t} + \psi_n(x, t) - \alpha \frac{\partial^2 \psi_n(x, t)}{\partial x^2} \right] = \\ &= \psi_n(x, \omega) - \frac{\omega}{1 - \beta\omega} Y \left[ \frac{\partial \psi_n(x, t)}{\partial t} \right] - \frac{\omega}{1 - \beta\omega} \left[ \psi_n(x, \omega) - \alpha \frac{\partial^2 \psi_n(x, \omega)}{\partial x^2} \right] \end{aligned} \quad (30)$$

with the initial condition:

$$\psi_0(x, \omega) = \omega \exp(x) \quad (31)$$

In a similar way, we have the first approximation:

$$\psi_1(x, \omega) = \exp(x) \left[ \frac{\alpha - 1}{\beta} \left( \frac{\omega}{1 - \beta\omega} - \omega \right) + \omega \right] \quad (32)$$

which deduces that:

$$\psi_1(x, t) = \exp(x) \left[ \frac{\alpha - 1}{\beta} (e^{\beta t} - 1) + 1 \right] \quad (33)$$

The first approximate solution of eq. (28) is given in fig. 4.

In a similar manner, we can give the second approximation:

$$\psi_2(x, \omega) = \exp(x) \left[ 1 + \frac{\alpha^2 - 3\alpha + 2}{2\beta^2} \left( \frac{\omega}{1 - 2\beta\omega} - \omega \right) + \left( \frac{2\alpha - 1}{\beta} - \frac{(\alpha - 1)^2}{\beta^2} \right) \left( \frac{\omega}{1 - \beta\omega} - \omega \right) \right] \quad (34)$$

which implies:

$$\psi_2(x, t) = \exp(x) \left[ 1 + \frac{\alpha^2 - 3\alpha + 2}{2\beta^2} (e^{2\beta t} - 1) + \left( \frac{2\alpha - 1}{\beta} - \frac{(\alpha - 1)^2}{\beta^2} \right) (e^{\beta t} - 1) \right] \quad (35)$$

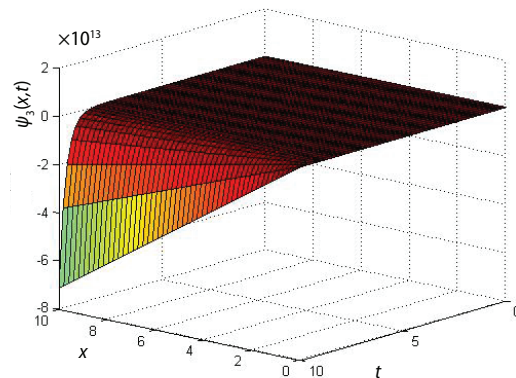
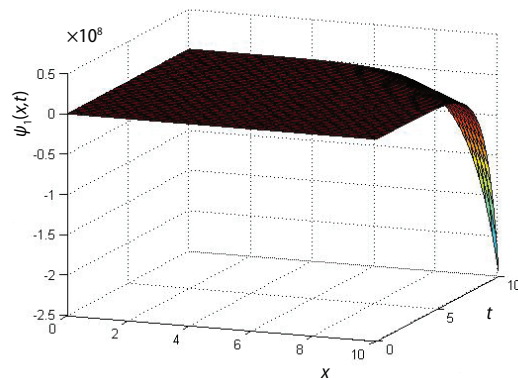
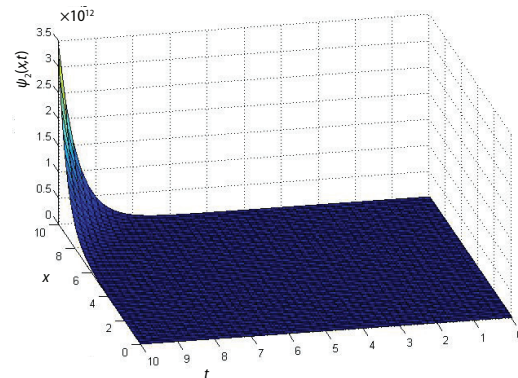


Figure 3. The plot of the approximate solution of eq. (18) after making the third iterative process (for color image see journal web site)

The second approximate solution of eq. (28) is illustrated in fig. 5.  
In a similar process, we obtain the other iterative solutions of eq. (28).



**Figure 4.** The plots of the first approximate solution of eq. (28) with the different parameters  $\alpha = 0.5$  and  $\beta = 1$   
(for color image see journal web site)



**Figure 5.** The plots of the second approximate solution of eq. (28) with the different parameters  $\alpha = 0.5$  and  $\beta = 1$   
(for color image see journal web site)

## Conclusion

In this work, we address new applications of the VIM integral transform method to handle the heat transfer equations at the low excess temperature. The analytical approximate solutions of the problems were graphically discussed in detail. The presented method is accurate and efficient for solving the heat transfer problems.

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## Nomenclature

$t$  – time co-ordinate, [s]  
 $x$  – space co-ordinate, [m]

### Greek symbols

$\alpha$  – thermal diffusivity, [ $\text{Wm}^{-1}\text{K}^{-1}$ ]  
 $\psi(x,t)$  – temperature, [K]

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