GENERAL FRACTIONAL-ORDER ANOMALOUS DIFFUSION WITH NON-SINGULAR POWER-LAW KERNEL

by

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In this paper, we investigate general fractional derivatives with a non-singular power-law kernel. The anomalous diffusion models with non-singular power-law kernel are discussed in detail. The results are efficient for modelling the anomalous behaviors within the frameworks of the Riemann-Liouville and Liouville-Caputo general fractional derivatives.

Key words: general fractional derivative with non-singular power-law kernel, Riemann-Liouville general fractional derivative, anomalous diffusion, Liouville-Caputo general fractional derivative

Introduction

The Riemann-Liouville and Liouville-Caputo fractional derivatives (FD) are known to have important roles in engineering applications, such as (for example) in heat transfer, viscoelasticity, and others, see [1-4] and the references cited therein. The theory of the Riemann-Liouville and Liouville-Caputo FD is used to model the anomalous diffusion behaviors. For example, the anomalous diffusion in the rotating flow was observed in [5]. The anomalous diffusion involving the stochastic pathway was discussed in [6]. The anomalous diffusion in the disordered (complex) media was reported in [7]. The anomalous diffusion with the external forces was presented in [8]. The anomalous diffusion related to the thermal equilibrium was considered in [9]. The anomalous diffusion in the sub diffusive case was proposed in [10].

Recently, the Riemann-Liouville and Liouville-Caputo general fractional derivatives (GFD) with non-singular Mittag-Leffler function kernels were introduced in [11] and their applications in the rheological models were discussed in [12]. More recently, the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel were presented in [13].
The Liouville-Weyl and Liouville-Caputo GFD with non-singular power-law kernel were proposed to model the anomalous diffusion problems in [14].

In the spirit of the previous ideas, the chief aim of this paper is to model general fractional anomalous diffusion problems with non-singular power-law kernel.

**Preliminary**

Let \( \mathbb{R}, \mathbb{R}_+, \mathbb{R}_-, \) and \( \mathbb{N} \) be the sets of the real numbers, positive real numbers, negative real numbers, and positive integer numbers, respectively.

In order to introduce the derivations of the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel, we start with the \( n \)-fold integral in the form, see [1]:

\[
\int_0^n \int_0^n \cdots \int_0^n \Theta(t) \, dt = \frac{1}{\Gamma(1+n)} \int_0^n (x-t)^n \Theta(t) \, dt
\]  

(1)

where \( n \in \mathbb{N}, x \in \mathbb{R}, \) and \( \Theta(t) \) is a real function.

From eq. (1) we have, see [1, 4]:

\[
\int_0^n \int_0^n \cdots \int_0^n \Theta(t) \, dt = \int_0^n \Lambda(x-t) \Theta(t) \, dt
\]  

(2)

where the kernel is represented in the form, see [1, 4]:

\[
\Lambda(x-t) = \frac{(x-t)^n}{\Gamma(1+n)}
\]  

(3)

Replacing \( n \) by \( \alpha \) in eq. (2), where \( \alpha \in \mathbb{R} \), we obtain:

\[
\Phi(x) = \int_0^n \frac{(x-t)^{\alpha}}{\Gamma(1+\alpha)} \Theta(t) \, dt
\]  

(4)

When \( \alpha = -\beta \in \mathbb{R}_- \), from eq. (4) we obtain the generalized Abel integral equation in the form [1]:

\[
\Phi(x) = \frac{1}{\Gamma(1-\beta)} \int_0^n \frac{\Theta(t)}{(x-t)^\beta} \, dt
\]  

(5)

The left-handed Riemann-Liouville FD of the function \( \Theta(t) \) of order \( \beta \) is defined as, see [1-4]:

\[
\mathcal{D}^{(\beta)}_{a} \Theta(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_a^x \frac{\Theta(t)}{(x-t)^\beta} \, dt
\]  

(6)

where \( a, \beta \in \mathbb{R} \) and \( 0 < \beta < 1 \).

The right-handed Riemann-Liouville FD of the function \( \Theta(t) \) of order \( \beta \) is defined as, see [1-4]:

\[
\mathcal{D}^{(\beta)}_{b} \Theta(x) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \int_x^b \frac{\Theta(t)}{(t-x)^\beta} \, dt
\]  

(7)
where $b, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The left-handed Riemann-Liouville FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [4]:

$$\mathcal{RL}_a^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(m - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x} \right)^{\beta - m + 1} dt$$  \hspace{1cm} (8)

where $a, \beta \in \mathbb{R}$ and $m - 1 < \beta < m$.

The right-handed Riemann-Liouville FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [4]:

$$\mathcal{RL}_b^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(m - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x - x} \right)^{\beta - m + 1} dt$$  \hspace{1cm} (9)

where $b, \beta \in \mathbb{R}$ and $m - 1 < \beta < m$.

The left-handed Liouville-Caputo FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [2-4]:

$$\mathcal{LC}_a^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x} \right)^{\beta - m + 1} dr$$  \hspace{1cm} (10)

where $a, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The right-handed Liouville-Caputo FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [2-4]:

$$\mathcal{LC}_b^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x - x} \right)^{\beta - m + 1} dr$$  \hspace{1cm} (11)

where $b, \beta \in \mathbb{R}$ and $0 < \beta < 1$.

The left-handed Liouville-Caputo FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [2-4]:

$$\mathcal{LC}_a^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(m - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x} \right)^{\beta - m + 1} dr$$  \hspace{1cm} (12)

where $a, \beta \in \mathbb{R}$ and $m - 1 < \beta < m$.

The right-handed Liouville-Caputo FD of the function $\Theta(t)$ of order $\beta$ is defined as, see [2-4]:

$$\mathcal{LC}_b^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(m - \beta)} \left( \frac{d}{dx} \right)^m \left( \frac{\Theta(t)}{x - x} \right)^{\beta - m + 1} dr$$  \hspace{1cm} (13)

where $b, \beta \in \mathbb{R}$ and $m - 1 < \beta < m$.

For $0 < \beta < 1$, one has [1, 4]:

$$\mathcal{RL}_a^\beta \frac{d}{dx} \Theta(x) = \frac{1}{\Gamma(1 - \beta)} \left( \frac{\Theta(a)}{x - a} \right)^{\beta - 1} + \mathcal{LC}_a^\beta \frac{d}{dx} \Theta(x)$$  \hspace{1cm} (14)
\[ R_{\alpha} L_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(1-\beta)} \frac{\Theta(b)}{(b-x)^\beta} - L_{\alpha} C_{\beta}^{(\rho)} \Theta(x) \]  

(15)

**Remark 1.** For more details of the Riemann-Liouville and Liouville-Caputo FD, readers refer to see [1-10].

**General fractional-order derivatives**

When \( \alpha = -\beta \in \mathbb{R}_+ \), from eq. (4) we obtain:

\[ \Phi(x) = \frac{1}{\Gamma(1+\beta)} \int_0^x (x-t)^\beta \Theta(t) \, dt \]  

(16)

The left-handed Riemann-Liouville GFD of the function \( \Theta(t) \) of order \( \beta \) is defined as [13]:

\[ R_{\alpha} L_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_a^x (x-t)^\beta \Theta(t) \, dt \]  

(17)

where \( a, \beta \in \mathbb{R} \) and \( 0 < \beta < 1 \).

The right-handed Riemann-Liouville GFD of the function \( \Theta(t) \) of order \( \beta \) is defined:

\[ R_{\alpha} L_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_a^b (t-x)^\beta \Theta(t) \, dt \]  

(18)

where \( b, \beta \in \mathbb{R} \) and \( 0 < \beta < 1 \).

The left-handed Riemann-Liouville GFD of the function \( \Theta(t) \) of order \( \beta \) is defined as [13]:

\[ R_{\alpha} L_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(m+\beta)} \left( \frac{d}{dx} \right)^{m+\beta} \int_a^x (x-t)^{\beta+m-1} \Theta(t) \, dt \]  

(19)

where \( a, \beta \in \mathbb{R} \) and \( m-1 < \beta < m \).

The right-handed Riemann-Liouville GFD of the function \( \Theta(t) \) of order \( \beta \) is defined:

\[ R_{\alpha} L_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(m+\beta)} \left( \frac{d}{dx} \right)^{m+\beta} \int_a^b (t-x)^{\beta+m-1} \Theta(t) \, dt \]  

(20)

where \( b, \beta \in \mathbb{R} \) and \( m-1 < \beta < m \).

The left-handed Liouville-Caputo GFD of the function \( \Theta(t) \) of order \( \beta \) is defined as, see [13]:

\[ L_{\alpha} C_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_a^x (x-t)^\beta \frac{d\Theta(t)}{dx} \, dt \]  

(21)

where \( a, \beta \in \mathbb{R} \) and \( 0 < \beta < 1 \).

The right-handed Liouville-Caputo GFD of the function \( \Theta(t) \) of order \( \beta \) is defined:

\[ L_{\alpha} C_{\beta}^{(\rho)} \Theta(x) = \frac{1}{\Gamma(1+\beta)} \frac{d}{dx} \int_a^b (t-x)^\beta \frac{d\Theta(t)}{dx} \, dt \]  

(22)

where \( b, \beta \in \mathbb{R} \) and \( 0 < \beta < 1 \).
The left-handed Liouville-Caputo GFD of the function \( \Theta(t) \) of order \( \beta \) is defined as,

\[
\text{GLC}_a^\beta \Theta(x) = \frac{1}{\Gamma(m+\beta)} \int_a^x (x-t)^{\beta-m-1} \frac{d^m \Theta(t)}{dx^m} dt
\]

where \( a, \beta \in \mathbb{R} \) and \( m-1 < \beta < m \).

The right-handed Liouville-Caputo GFD of the function \( \Theta(t) \) of order \( \beta \) is defined as:

\[
\text{GLC}_b^\beta \Theta(x) = \frac{1}{\Gamma(m+\beta)} \int_x^b (t-x)^{\beta-m-1} \frac{d^m \Theta(t)}{dx^m} dt
\]

where \( b, \beta \in \mathbb{R} \) and \( m-1 < \beta < m \).

For \( 0 < \beta < 1 \), we obtain:

\[
\text{GLC}_a^\beta \Theta(x) = \frac{(x-a)^\beta \Theta(a)}{\Gamma(1-\beta)} + \text{GLC}_b^\beta \Theta(x)
\]

\[
\text{GLC}_b^\beta \Theta(x) = \frac{(b-x)^\beta \Theta(b)}{\Gamma(1-\beta)} - \text{GLC}_b^\beta \Theta(x)
\]

Remark 2. For more details of the definitions of the left-handed Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law and Mittag-Leffler-function kernels, readers refer to [11-18].

The Laplace transforms of eqs. (18) and (21) are as follows [13]:

\[
L\left[ \left( \text{GLC}_a^\beta \Theta \right)(x) \right] = \frac{1}{s^\beta} \Theta(s)
\]

\[
L\left[ \left( \text{GLC}_0^\beta \Theta \right)(x) \right] = \frac{1}{s^\beta} \left[ s\Theta(s) - \Theta(0) \right]
\]

where the Laplace transform is defined by [1, 4]:

\[
L\left[ \Phi (x) \right] = \Phi(s) = \int_0^\infty e^{-sx} \Phi(x) dx
\]

Remark 3. For more details of the definitions of the left-handed Riemann-Liouville and Liouville-Caputo FD and GFD, readers refer to [1-22].

New results

Let us consider the following expressions of the GFD with non-singular power-law kernel:

\[
\text{GLC}_a^\beta \Theta(x) = \frac{1}{\Gamma(1+i\beta)} \frac{d}{dx} \int_a^x (x-t)^i \Theta(t) dt
\]

\[
\text{GLC}_b^\beta \Theta(x) = \frac{1}{\Gamma(1+i\beta)} \frac{d}{dx} \int_x^b (t-x)^i \Theta(t) dt
\]
\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{1}{\Gamma(1+i\beta)} \int_{a}^{b} (x-t)^{i\beta} \frac{d\Theta(t)}{dx} dt
\]  
(32)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{1}{\Gamma(1+i\beta)} \int_{a}^{b} (t-x)^{i\beta} \frac{d\Theta(t)}{dx} dt
\]  
(33)

For 0 < \beta < 1, we have the following GFD with non-singular power-law kernel:

\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} (x-t)^{i\beta} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[(x-t)^{\beta}] \Theta(t) dr
\]  
(34)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[(t-x)^{\beta}] \Theta(t) dr
\]  
(35)

\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} (-1)^{i} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[-(t-x)^{\beta}] \Theta(t) dr
\]  
(36)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} (-1)^{i} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[-(t-x)^{\beta}] \Theta(t) dr
\]  
(37)

\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} (x-t)^{i\beta} \frac{d\Theta(t)}{dr} dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[(x-t)^{\beta}] \frac{d\Theta(t)}{dr} dr
\]  
(38)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} (-1)^{i} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \frac{d\Theta(t)}{dr} dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[-(t-x)^{\beta}] \frac{d\Theta(t)}{dr} dr
\]  
(39)

\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \frac{d\Theta(t)}{dr} dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[(t-x)^{\beta}] \frac{d\Theta(t)}{dr} dr
\]  
(40)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} (-1)^{i} \frac{1}{\Gamma(1+i\beta)} (t-x)^{i\beta} \frac{d\Theta(t)}{dr} dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[-(t-x)^{\beta}] \frac{d\Theta(t)}{dr} dr
\]  
(41)

where the Mittag-Leffler function is defined as in [1]:

\[
E_{\beta}[(x-t)^{\beta}] = \sum_{i=0}^{\infty} \frac{(x-t)^{i\beta}}{\Gamma(1+i\beta)}
\]

In a similar way, from eqs. (17), (18), (21), and (22) we find for 0 < \beta < 1 that:

\[
\frac{\text{GLC}}{x}D_{\alpha}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} \frac{1}{\Gamma(1-i\beta)} \frac{1}{(x-t)^{i\beta}} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[(x-t)^{\beta}] \Theta(t) dr
\]  
(42)

\[
\frac{\text{GLC}}{x}D_{\beta}^{(\beta)} \Theta(x) = \frac{d}{dx} \int_{a}^{b} \sum_{i=0}^{\infty} (-1)^{i} \frac{1}{\Gamma(1-i\beta)} (x-t)^{i\beta} \Theta(t) dr = \frac{d}{dx} \int_{a}^{b} E_{\beta}[-(x-t)^{\beta}] \Theta(t) dr
\]  
(43)
Modelling the general fractional anomalous diffusion with non-singular power-law kernel

We now consider the Riemann-Liouville general fractional time anomalous diffusion with non-singular power-law kernel:

\[
GRL_{\alpha, \beta}^{(\beta)} \Theta(x, \tau) = \frac{1}{\Gamma(1 + \beta)} \frac{d}{d\tau} \left[ \int_{0}^{\tau} (\tau - \tau')^{\beta} \Theta(x, \tau') d\tau' \right] \]

subject to the initial condition

\[
\Theta(x, 0) = g(x) \]

where \( \kappa \) is the diffusion coefficient and the Riemann-Liouville general fractional partial derivative of the function \( \Theta(x, \tau) \) of order \( \beta \) with respect to the time variable, \( \tau \), is defined by:

\[
GRL_{\alpha, \beta}^{(\beta)} \Theta(x, \tau) = \frac{1}{\Gamma(1 + \beta)} \frac{d}{d\tau} \left[ \int_{0}^{\tau} (\tau - \tau')^{\beta} \Theta(x, \tau') d\tau' \right] \]

Let us consider the Liouville-Caputo general fractional time anomalous diffusion with non-singular power-law kernel:

\[
GLC_{\alpha, \beta}^{(\beta)} \Theta(x, \tau) = \frac{1}{\Gamma(1 + \beta)} \frac{d}{d\tau} \left[ \int_{0}^{\tau} (\tau - \tau')^{\beta} \Theta(x, \tau') d\tau' \right] \]

subject to the initial condition
\[ \Theta(x,0) = \delta(x) \]  

(54)

where \( \delta(x) \) is the Dirac delta function \([4]\) and the Liouville-Caputo general fractional partial derivative of the function \( \Theta(x, \tau) \) of order \( \beta \) with respect to the time variable, \( \tau \), is defined by:

\[
\frac{d^{\beta}}{dt^{\beta}} \Theta(x, \tau) = \frac{1}{\Gamma(1+\beta)} \int_0^\tau (\tau - s)^{\beta-1} \frac{d\Theta(x,s)}{ds} \, ds 
\]

(55)

Conclusion

The present study addressed the derivations of the Riemann-Liouville and Liouville-Caputo GFD with non-singular power-law kernel. The relationship between the GFD with non-singular power-law and Mittag-Leffler function kernels were discussed. The Riemann-Liouville and Liouville-Caputo general fractional time anomalous diffusion models with non-singular power-law kernel were obtained. The models are successfully adopted to model the anomalous behaviors of the complex phenomena.

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Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t )</td>
<td>time co-ordinate, [s]</td>
<td></td>
</tr>
<tr>
<td>( x )</td>
<td>space co-ordinate, [m]</td>
<td></td>
</tr>
</tbody>
</table>

Greek Symbols

- \( \beta \) – fractional order, [-]
- \( \kappa \) – diffusion coefficient, \([m^2\cdot s^{-1}]\)

References