GENERAL FRACTIONAL CALCULUS IN NON-SINGULAR POWER-LAW KERNEL APPLIED TO MODEL ANOMALOUS DIFFUSION PHENOMENA IN HEAT TRANSFER PROBLEMS

by

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Original scientific paper
https://doi.org/10.2298/TSCI170310194G

In this paper we address the general fractional calculus of Liouville-Weyl and Liouville-Caputo general fractional derivative types with non-singular power-law kernel for the first time. The Fourier transforms and the anomalous diffusions are discussed in detail. The formulations are adopted to describe complex phenomena of the heat transfer problems.

Key words: *heat transfer, anomalous diffusion, general fractional calculus, Fourier transforms*

Introduction

As one of important branches of fractional derivatives (FD) [1-5], the general fractional derivatives (GFD) have played an important role in being applied in mathematics and physics, see [6-10] and the cited references therein. For instance, the general evolution equation involving the general fractional calculus (GFC) was discussed in [11]. The general time fractional diffusion equation was reported in [12]. The rheological problems involving the GFD within the non-singular power-law [13] and Mittag-Leffler-function [14] kernels were proposed, respectively. The GFD within the extended Mittag-Leffler type functions were also reported in [15]. The GFD with a constant from the normalized process were proposed in [16, 17].

However, the GFC of the Liouville-Weyl and Liouville-Caputo type with the non-singular power-law kernel have not yet reported anywhere. With a strong motivation of the ideas aforementioned, the brief objective of the present paper is to suggest general Liouville-Weyl and Liouville-Caputo GFD in the non-singular power-law kernel and to model the anomalous diffusion phenomena in the heat transfer problems.

The Liouville-Weyl GFC in non-singular power-law kernel

In this section, we first consider the Liouville-Weyl fractional calculus in singular power-law kernel. With the use of them, the GFC in the non-singular power-law kernel is proposed.

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The Liouville-Weyl and Liouville-Caputo FD within singular power-law kernel

Let $\mathbb{R}$ and $\mathbb{N}$ be sets of real numbers and positive integral numbers, respectively, and let $\Omega \in C[a,b]$, where $a,b \in \Re$.

The left-side Liouville-Weyl FD of the function $\Omega(t)$ of order α is defined by [6, 18, 19]:

$$\left[{}_{-\infty}^{\mathrm{LW}}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{t}(t-\eta)^{-\alpha}\,\Omega(\eta)\,\mathrm{d}\eta =$$
$$=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\left[\Omega(t)-\Omega(t+\eta)\right]\eta^{-\alpha-1}\mathrm{d}\eta \tag{1}$$

where $-\infty < t$ and $0 < \alpha < 1$.

The right-side Liouville-Weyl FD of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{t}^{\mathrm{LW}}\mathbb{D}_{\infty}^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{t}^{\infty}(t-\eta)^{-\alpha}\,\Omega(\eta)\,\mathrm{d}\eta =$$
$$=\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{\infty}\left[\Omega(t)-\Omega(t+\eta)\right]\eta^{-\alpha-1}\mathrm{d}\eta \tag{2}$$

where $t < \infty$ and $0 < \alpha < 1$.

The left-side Liouville-Weyl FD of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{-\infty}^{\mathrm{LW}}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^{t}(t-\eta)^{n-\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{3}$$

where $-\infty < t$, $n-1 < \alpha < n$ and $n \in \mathbb{N}$.

The right-side Liouville-Weyl FD of the function $\Omega(t)$ of order α is defined by [6, 18, 19]:

$$\left[{}_{t}^{\mathrm{LW}}\mathbb{D}_{\infty}^{(\alpha)}\Omega\right](t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{t}^{\infty}(t-\eta)^{n-\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{4}$$

where $t < \infty$, $n-1 < \alpha < n$ and $n \in \mathbb{N}$.

The left-side Liouville-Weyl fractional integral of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{-\infty}^{\mathrm{LW}}\mathbb{I}_{\tau}^{(\alpha)}\Omega\right](\tau)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{\tau}(t-\mu)^{\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{5}$$

where $-\infty < t$ and $0 < \alpha$.

The left-side Liouville-Weyl fractional integral of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{t}^{\mathrm{LW}}\mathbb{I}_{+\infty}^{(\alpha)}\Omega\right](\tau)=\frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(t-\eta)^{\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{6}$$

where $t < \infty$ and $0 < \alpha$.

The left-side Liouville-Caputo FD of the function $\Omega(t)$ of order α is defined by [6, 18, 19]:

$$\left[{}_{-\infty}^{\mathrm{LC}}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-\eta)^{-\alpha}\frac{\mathrm{d}\Omega(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{7}$$

where $-\infty < t$ and $0<\alpha<1$.

The right-side Liouville-Caputo FD of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{t}^{\mathrm{LC}}\mathbb{D}_\infty^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1-\alpha)}\int_{t}^{\infty}(t-\eta)^{-\alpha}\frac{\mathrm{d}\Omega(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{8}$$

where $t<\infty$ and $0<\alpha<1$.

The left-side Liouville-Caputo FD of the function $\Omega(t)$ of order α is defined as [6, 18, 19]:

$$\left[{}_{-\infty}^{\mathrm{LC}}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(n-\alpha)}\int_{-\infty}^{t}(t-\eta)^{n-\alpha-1}\frac{\mathrm{d}^n\Omega(\eta)}{\mathrm{d}\eta^n}\mathrm{d}\eta \tag{9}$$

where $-\infty<t$, $n-1<\alpha<n$ and $n\in\mathbb{N}$.

The right-side Liouville-Caputo FD of the function $\Omega(t)$ of order α is defined by [6, 18, 19]:

$$\left[{}_{t}^{\mathrm{LC}}\mathbb{D}_\infty^{(\alpha)}\Omega\right](t)=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_{t}^{\infty}(t-\eta)^{n-\alpha-1}\frac{\mathrm{d}^n\Omega(\eta)}{\mathrm{d}\eta^n}\mathrm{d}\eta \tag{10}$$

where $t<\infty$, $n-1<\alpha<n$ and $n\in\mathbb{N}$.

The Riemann-Liouville and Liouville-Caputo GFD within non-singular power-law kernel

The Riemann-Liouville GFD of the function $\Omega(t)$ of order α is defined by [13]:

$$\left[{}_{0}^{\mathrm{LRT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{0}^{t}(\eta-t)^{\alpha}\,\Omega(\eta)\mathrm{d}\eta \tag{11}$$

where $\eta>0$ and $0<\alpha<1$.

The Riemann-Liouville GFD of the function $\Omega(t)$ of order α is defined by [13]:

$$\left[{}_{0}^{\mathrm{LRT}}\mathbb{D}_\tau^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(n+\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{0}^{\tau}(t-\eta)^{n+\alpha-1}\,\Omega(\eta)\mathrm{d}\eta \tag{12}$$

where $0<t$, $n-1<\alpha<n$ and $n\in\mathbb{N}$.

The Riemann-Liouville fractional integral (GFI) of the function $\Omega(t)$ of order α is defined as [13]:

$$\left[{}_{t}^{\mathrm{LRT}}\mathbb{I}_{+\infty}^{(\alpha)}\Omega\right](\tau)=\frac{1}{\Gamma(-\alpha)}\int_{0}^{t}(t-\eta)^{-\alpha-1}\,\Omega(\eta)\mathrm{d}\eta \tag{13}$$

where $t<\infty$ and $0<\alpha$.

The Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined by [13]:

$$\left[{}^{\mathrm{LCT}}_{\ 0}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\int_0^t(\eta-t)^{\alpha}\frac{\mathrm{d}\Omega(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{14}$$

where $\eta > 0$ and $0 < \alpha < 1$.

The Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined by [13]:

$$\left[{}^{\mathrm{LCT}}_{\ 0}\mathbb{D}_\tau^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(n+\alpha)}\int_0^\tau(t-\eta)^{n+\alpha-1}\frac{\mathrm{d}^n\Omega(\eta)}{\mathrm{d}\eta^n}\mathrm{d}\eta \tag{15}$$

where $0 < t$, $n-1 < \alpha < n$ and $n \in \mathbb{N}$.

The Liouville-Weyl and Liouville-Caputo GFD within non-singular power-law kernel

As the results of the generalizations of the Riemann-Liouville and Liouville-Caputo GFD within non-singular power-law kernel, we now consider the definitions of the Liouville-Weyl and Liouville-Caputo GFD in the real line.

The left-side Liouville-Weyl GFD of the function $\Omega(t)$ of order α is defined by:

$$\left[{}^{\mathrm{LWT}}_{-\infty}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^t(t-\eta)^{\alpha}\Omega(\eta)\mathrm{d}\eta=$$
$$=\frac{\alpha}{\Gamma(1+\alpha)}\int_0^\infty\left[\Omega(t+\eta)-\Omega(t)\right]\eta^{\alpha-1}\mathrm{d}\eta \tag{16}$$

where $-\infty < t$ and $0 < \alpha < 1$.

The right-side Liouville-Weyl GFD of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LWT}}_{\ t}\mathbb{D}_\infty^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty(t-\eta)^{\alpha}\Omega(\eta)\mathrm{d}\eta=$$
$$=\frac{\alpha}{\Gamma(1+\alpha)}\int_0^\infty\left[\Omega(t+\eta)-\Omega(t)\right]\eta^{\alpha-1}\mathrm{d}\eta \tag{17}$$

where $t < \infty$ and $0 < \alpha < 1$.

The left-side Liouville-Weyl GFD of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LWT}}_{-\infty}\mathbb{D}_t^{(\alpha)}\Omega\right](t)=\frac{1}{\Gamma(n+\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_{-\infty}^t(t-\eta)^{n+\alpha-1}\Omega(\eta)\mathrm{d}\eta \tag{18}$$

where $-\infty < t$, $n-1 < \alpha < n$, and $n \in \mathbb{N}$.

The right-side Liouville-Weyl GFD of the function $\Omega(t)$ of order α is defined by:

$$\left[{}^{\mathrm{LWT}}_{\ t}\mathbb{D}_\infty^{(\alpha)}\Omega\right](t)=\frac{(-1)^n}{\Gamma(n+\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_t^\infty(t-\eta)^{n+\alpha-1}\Omega(\eta)\mathrm{d}\eta \tag{19}$$

where $t < \infty$, $n-1 < \alpha < n$, and $n \in \mathbb{N}$.

The left-side Liouville-Weyl GFI of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LWT}}_{-\infty}\mathbb{I}^{(\alpha)}_{t}\Omega\right](t)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{t}(t-\eta)^{-\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{20}$$

where $-\infty < t$ and $0 < \alpha$.

The left-side Liouville-Weyl GFI of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LWT}}_{t}\mathbb{I}^{(\alpha)}_{\infty}\Omega\right](t)=\frac{1}{\Gamma(-\alpha)}\int_{t}^{\infty}(t-\eta)^{-\alpha-1}\,\Omega(\eta)\,\mathrm{d}\eta \tag{21}$$

where $t < \infty$ and $0 < \alpha$.

The left-side Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined by:

$$\left[{}^{\mathrm{LCT}}_{-\infty}\mathbb{D}^{(\alpha)}_{t}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{t}(t-\eta)^{\alpha}\,\frac{\mathrm{d}\Omega(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{22}$$

where $-\infty < t$ and $0 < \alpha < 1$.

The right-side Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LCT}}_{t}\mathbb{D}^{(\alpha)}_{\infty}\Omega\right](t)=\frac{1}{\Gamma(1+\alpha)}\int_{t}^{\infty}(t-\eta)^{\alpha}\,\frac{\mathrm{d}\Omega(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{23}$$

where $t < \infty$ and $0 < \alpha < 1$.

The left-side Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined:

$$\left[{}^{\mathrm{LCT}}_{-\infty}\mathbb{D}^{(\alpha)}_{t}\Omega\right](t)=\frac{1}{\Gamma(n+\alpha)}\int_{-\infty}^{t}(t-\eta)^{n+\alpha-1}\,\frac{\mathrm{d}^{n}\Omega(\eta)}{\mathrm{d}\eta^{n}}\mathrm{d}\eta \tag{24}$$

where $-\infty < t$, $n-1 < \alpha < n$, and $n \in \mathbb{N}$.

The right-side Liouville-Caputo GFD of the function $\Omega(t)$ of order α is defined by:

$$\left[{}^{\mathrm{LCT}}_{t}\mathbb{D}^{(\alpha)}_{\infty}\Omega\right](t)=\frac{(-1)^{n}}{\Gamma(n+\alpha)}\int_{t}^{\infty}(t-\eta)^{n+\alpha-1}\,\frac{\mathrm{d}^{n}\Omega(\eta)}{\mathrm{d}\eta^{n}}\mathrm{d}\eta \tag{25}$$

where $t < \infty$, $n-1 < \alpha < n$, and $n \in \mathbb{N}$.

The Fourier transform of the function is defined:

$$\mathbb{F}\left[\Omega(t)\right](\omega):=\Omega(\omega)=\int_{-\infty}^{\infty}\exp^{-it\omega}\,\Omega(t)\,\mathrm{d}t \tag{26}$$

where $\omega, t \in \Re$.

In tab. 1, the expression $(i\omega)^{\alpha}$ is defined by [6, 19]:

$$(i\omega)^{\alpha}=|\omega|^{\alpha}\exp(\alpha\pi i)\frac{\mathrm{sgn}(\omega)}{2} \tag{27}$$

where $\mathrm{sgn}(\omega)$ is the sign function [20].

The Fourier transforms (FT) of the Liouville-Weyl GFD, GFI, and Liouville-Caputo GFD are listed in tab. 1.

Table 1. The FT of the Liouville-Weyl GFD, GFI, and Liouville-Caputo GFD

GFD	FT
$\left[{}^{\mathrm{LWT}}_{-\infty}\mathbb{D}^{(\alpha)}_{t}\Omega\right](t)$	$(i\omega)^{-\alpha}\,\Omega(\omega)$
$\left[{}^{\mathrm{LWT}}_{t}\mathbb{D}^{(\alpha)}_{\infty}\Omega\right](t)$	$(-i\omega)^{-\alpha}\,\Omega(\omega)$
$\left[{}^{\mathrm{LWT}}_{-\infty}\mathbb{I}^{(\alpha)}_{t}\Omega\right](t)$	$(i\omega)^{\alpha}\,\Omega(\omega)$
$\left[{}^{\mathrm{LWT}}_{t}\mathbb{I}^{(\alpha)}_{\infty}\Omega\right](t)$	$(-i\omega)^{\alpha}\,\Omega(\omega)$
$\left[{}^{\mathrm{LCT}}_{-\infty}\mathbb{D}^{(\alpha)}_{t}\Omega\right](t)$	$(i\omega)^{-\alpha}\,\Omega(\omega)$
$\left[{}^{\mathrm{LCT}}_{t}\mathbb{D}^{(\alpha)}_{\infty}\Omega\right](t)$	$(-i\omega)^{-\alpha}\,\Omega(\omega)$

Modelling anomalous diffusion phenomena involving GFD within non-singular power-law kernel

We now consider the general Liouville-Weyl time fractional anomalous diffusion (subdiffusion) within non-singular power-law kernel:

$$\left[{}_{-\infty}^{\mathrm{LWT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t,x)=\kappa\frac{\partial^2\Omega(t,x)}{\partial x^2},\quad 0<\alpha<1 \tag{28}$$

where κ is the diffusion coefficient, x is the space, and t is the time, and:

$$\left[{}_{-\infty}^{\mathrm{LWT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t,x)=\frac{1}{\Gamma(1+\alpha)}\frac{\partial}{\mathrm{d}\eta}\int_{-\infty}^{t}(t-\eta)^{\alpha}\,\Omega(\eta,x)\,\mathrm{d}\eta \tag{29}$$

Let us consider the general Liouville-Caputo time fractional anomalous diffusion (subdiffusion) within non-singular power-law kernel:

$$\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t,x)=\kappa\frac{\partial^2\Omega(t,x)}{\partial x^2},\quad 0<\alpha<1 \tag{30}$$

where κ is the diffusion coefficient, x – the space, and t – the time, and:

$$\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t,x)=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{t}(t-\eta)^{\alpha}\,\frac{\partial\Omega(\eta,x)}{\mathrm{d}\eta}\mathrm{d}\eta \tag{31}$$

The general Liouville-Weyl fractional-space anomalous diffusion (superdiffusion) within non-singular power-law kernel:

$$\frac{\partial\Omega(t,x)}{\partial t}=\kappa\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_x^{(\alpha)}\Omega\right](t,x),\quad 1<\alpha<2 \tag{32}$$

is presented, where κ is the diffusion coefficient, x – the space, and t – the time, and:

$$\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_x^{(\alpha)}\Omega\right](t,x)=\frac{1}{\Gamma(2+\alpha)}\frac{\partial^2}{\mathrm{d}\eta^2}\int_{-\infty}^{x}(x-\eta)^{\alpha+1}\,\Omega(t,\eta)\,\mathrm{d}\eta \tag{33}$$

The general Liouville-Caputo fractional-space anomalous diffusion (superdiffusion) within non-singular power-law kernel:

$$\frac{\partial\Omega(t,x)}{\partial t}=\kappa\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_x^{(\alpha)}\Omega\right](t,x),\quad 1<\alpha<2 \tag{34}$$

is considered, where κ is the diffusion coefficient, x – the space, and t – the time, and:

$$\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_x^{(\alpha)}\Omega\right](t,x)=\frac{1}{\Gamma(2+\alpha)}\int_{-\infty}^{x}(x-\eta)^{\alpha+1}\,\frac{\partial^2\Omega(t,\eta)}{\mathrm{d}\eta^2}\mathrm{d}\eta \tag{35}$$

Similarly, the general Liouville-Caputo time fractional anomalous diffusion (superdiffusion) within non-singular power-law kernel:

$$\left[{}_{-\infty}^{\mathrm{LCT}}\mathbb{D}_t^{(\alpha)}\Omega\right](t,x)=\kappa\frac{\partial^2\Omega(t,x)}{\partial x^2},\quad 1<\alpha<2 \tag{36}$$

is considered, where κ is the diffusion coefficient, x – the space, and t – the time.

Conclusion

In the present work, we proposed the Riemann-Liouville and Liouville-Caputo GFD within non-singular power-law kernel in the real line for the first time. The anomalous diffusions were discussed with the use of the proposed general fractional-order differential operators. The general fractional-order differential subdiffusion and super diffusion equations involving Liouville-Caputo GFD within non-singular power-law kernel were also presented. The proposed formulations are presented as a new prospective for describing the anomalous heat transfer problems.

Acknowledgment

This work is supported by the State Key Research Development Program of the People Republic of China (Grant No. 2016YFC0600705), the Natural Science Foundation of China (Grant No.51323004), and the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD2014).

Nomenclature

x – space co-ordinate, [m]
t – time co-ordinate, [s]

Greek symbols

α – fractional order, [–]
κ – diffusion coefficient, [m^2s^{-1}]
$\Omega(\omega)$ – Fourier transform of $\Omega(t)$, [–]

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Paper submitted: March 10, 2017
Paper revised: May 1, 2017
Paper accepted: May 20, 2017