ON LINEAR VISCOELASTICITY WITHIN GENERAL FRACTIONAL DERIVATIVES WITHOUT SINGULAR KERNEL

by

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The Riemann-Liouville and Caputo-Liouville fractional derivatives without singular kernel are proposed as mathematical tools to describe the mathematical models in line viscoelasticity in the present article. The fractional mechanical models containing the Maxwell and Kelvin-Voigt elements are graphically discussed with the Laplace transform. The results are accurate and efficient to reveal the complex behaviors of the real materials.

Key words: *fractional derivatives without singular kernel, Laplace transform, Maxwell element, Kelvin-Voigt element, linear viscoelasticity*

Introduction

Fractional derivatives (FD) of the Riemann-Liouville (RL) and Liouville-Caputo (LC) types [1-5] due to the power-law functions has been successfully utilized to investigate the important models in the fields of solid mechanics, especially in linear viscoelasticity (see, *e. g.*, [6-11] and the related references therein). The general FD involving the kernels of the exponential and Mittag-Leffler functions were proposed in the sense of the RL and LC types, see [12-21]. For example, a new fractional-order heat-diffusion equation (FHDE) was proposed in [12]. The fractional-order groundwater flow (FGF) was reported in [13]. The fractional-order RLC circuit (FRLCC) was suggested in [14]. The heat-flows (HF) of fractional-order were investigated in [15-17]. The anomalous diffusion [18] and relaxation [19] models within general FD were proposed. The rheological models in the different operators were reported in [20-22].

Recently, the general FD of Caputo-Fabrizio type was used to model the fractional-order Maxwell model (see, *e. g.*, [23]). The main aim of the previous article is to investigate the fractional-order Maxwell and Kelvin-Voigt models in the general FD with the aid of the Laplace transform.

Mathematical tools

In this section, we present the definitions of the general FD without singular kernel in the sense of the RL and LC types used in this paper.

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The general FD without singular kernel of the function $\psi(t)$ in the LC sense, proposed by Caputo and Fabrizio, is defined as, see [1, 12-18]: $p(\upsilon) = \upsilon/(1-\upsilon)$, and $\mathrm{M}(\upsilon)$ represents (see [23] and the cited references therein):

$$ {}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\psi(t) = \mathrm{M}(\upsilon)\int_0^t \mathrm{e}^{[-p(\upsilon)(t-\tau)]}\psi^{(1)}(\tau)\,\mathrm{d}\tau \tag{1} $$

where $\Lambda(\upsilon)$ is a normalization parameter with respect to υ ($\upsilon \in (0,1]$) and $p(\upsilon) = \upsilon/(1-\upsilon)$, and $\mathrm{M}(\upsilon)$ represents (see [23] and the cited references therein).

$$ \mathrm{M}(\upsilon) = \frac{(2-\upsilon)\Lambda(\upsilon)}{2(1-\upsilon)} = \frac{1}{1-\upsilon} \tag{2} $$

The general FD without singular kernel of the function $\psi(t)$ in the sense of the RL type, proposed by Yang, Srivastava and Machado, is defined as [15, 18]:

$$ {}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\psi(t) = \mathrm{M}(\upsilon)\frac{\partial}{\partial t}\int_0^t \mathrm{e}^{[-p(\upsilon)(t-\tau)]}\psi(\tau)\,\mathrm{d}\tau \tag{3} $$

The relationship of eqs. (1) and (3) is presented as [13]:

$$ {}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\psi(t) = {}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\psi(t) - \mathrm{M}(\upsilon)e^{(-t)}\psi(0) \tag{4} $$

The general FD without singular kernel of the function $\psi(t)$ in the LC sense is defined as, see [8]:

$$ {}_{e(-)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t) = \int_0^t \mathrm{e}^{[-(t-\tau)]}\psi^{(1)}(\tau)\,\mathrm{d}\tau \tag{5} $$

and the general FD without singular kernel of the function $\psi(t)$ in the RL sense as [18]:

$$ {}_{e(-)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t) = \frac{\partial}{\partial t}\int_0^t \mathrm{e}^{[-(t-\tau)]}\psi(\tau)\,\mathrm{d}\tau \tag{6} $$

The relationship of eqs. (5) and (6) is as follows [18]:

$$ {}_{e(-)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t) = {}_{e(-)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t) - e^{-t}\psi(0) \tag{7} $$

The general FD without singular kernel of the function $\psi(t)$ in the LC sense is defined as, see [8]:

$$ {}_{e(+)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t) = \int_0^t \mathrm{e}^{(t-\tau)}\psi^{(1)}(\tau)\,\mathrm{d}\tau \tag{8} $$

and the general FD without singular kernel of the function $\psi(t)$ in the RL sense by [18]:

$$ {}_{e(+)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t) = \frac{\partial}{\partial t}\int_0^t \mathrm{e}^{(t-\tau)}\psi(\tau)\,\mathrm{d}\tau \tag{9} $$

The relationship of eqs. (8) and (9) is as follows [18]:

$$ {}_{e(+)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t) = {}_{e(+)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t) - e^{t}\psi(0) \tag{10} $$

The general FD without singular kernel of the function $\psi(t)$ in the LC sense is defined:

$$ {}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t)=\int_0^t \frac{\psi^{(1)}(\tau)}{t-\tau}\mathrm{d}\tau \tag{11}$$

and the general FD without singular kernel of the function $\psi(t)$ in the RL sense as:

$$ {}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t)=\frac{\partial}{\partial t}\int_0^t \frac{\psi(\tau)}{t-\tau}\mathrm{d}\tau \tag{12}$$

The relationship of eqs. (11) and (12) is as:

$$ {}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t)={}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t)-\frac{1}{t}\psi(0) \tag{13}$$

We notice that eqs. (12) and (13) are the special cases where the orders of the RL and CL derivatives is equal to 1.

The Laplace transform (LT) of the function $\psi(t)$ is defined as [18]:

$$ \mathbb{F}\left[\psi(t)\right]=\psi(s):=\int_0^\infty \mathrm{e}^{-st}\psi(t)\mathrm{d}t \tag{14}$$

The LT of the general FD operators [18] are listed in tab. 1.

Table 1. The LT of the different general FD operators

General FD operators	LT
${}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\psi(t)$	$\frac{\mathrm{M}(\upsilon)}{s+p(\upsilon)}\left[s\psi(s)-\psi(0)\right]$
${}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\psi(t)$	$\frac{\mathrm{M}(\upsilon)s}{s+p(\upsilon)}\psi(s)$
${}_{e(-)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t)$	$\frac{1}{s+1}\left[s\psi(s)-\psi(0)\right]$
${}_{e(-)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t)$	$\frac{s\psi(s)}{s+1}$
${}_{e(+)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t)$	$\frac{1}{s-1}\left[s\psi(s)-\psi(0)\right]$
${}_{e(+)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t)$	$\frac{s\psi(s)}{s-1}$
${}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\psi(t)$	$s\left[s\psi(s)-\psi(0)\right]$
${}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\psi(t)$	$s^2\psi(s)$

Fractional mechanical models

In this section, we discuss the laws of deformation for the real material and general mechanical models.

Laws of deformation for the real material

According to the equation of the power-law stress relaxations [21], we easily write the equations of the stress relaxation:

$$ \sigma(t)=\gamma t^{-1}\varepsilon(\tau),\quad \sigma(t)=\gamma t\varepsilon(t),\quad \sigma(t)=\gamma\varepsilon(t)\mathrm{e}^{-t},\quad \sigma(t)=\gamma\ \mathrm{e}^{t}\varepsilon(t) \tag{15a,b,c,d}$$

where $\sigma(t)$ is the stress, and $\varepsilon(t)$ is the strain.

We notice that eq. (15a) is the classical case [7] and do not discuss it here.

Fractional mechanical models

In order to present the general fractional-order Maxwell and Kelvin-Voigt models, we consider the following general fractional mechanical elements containing spring and dashpot models:

$$ \sigma(\tau)=\gamma\varepsilon(\tau) \tag{16}$$

$$ \sigma(\tau)=\Xi\frac{\mathrm{d}^\upsilon\varepsilon(t)}{\mathrm{d}t^\upsilon} \tag{17}$$

respectively, where γ is the Young's modulus, Ξ is the viscosity, υ is the material-dependant parameter, and $\partial^\upsilon/\partial t^\upsilon$ represents the general FD operators mentioned above. Expressions

$J(t)=\varepsilon(t)/\sigma(0)$ and $G(t)=\sigma(t)/\varepsilon(0)$ are the creep compliance function (CCF) and relaxation modulus function (RMF), where $\sigma(0)$ is the initial stress, and $\varepsilon(0)$ is the initial strain.

With the help of the Boltzmann superposition principle and under the assumption of causal histories [7, 23], we have the following constitutive equations of stress and strain of the Volterra type:

$$\varepsilon(t)=\sigma(0)J(t)+\int_0^t J(t-\tau)\frac{\mathrm{d}^{\upsilon}\sigma(\tau)}{\mathrm{d}\tau^{\upsilon}}\mathrm{d}\tau \tag{18}$$

and

$$\sigma(t)=\varepsilon(0)G(t)+\int_0^t G(t-\tau)\frac{\mathrm{d}^{\upsilon}\varepsilon(\tau)}{\mathrm{d}\tau^{\upsilon}}\mathrm{d}\tau \tag{19}$$

respectively.

Taking the Laplace transform for eqs. (18) and (19) yields:

$$\varepsilon(s)=J(s)E(s)\sigma(s),\quad \sigma(s)=G(s)E(s)\varepsilon(s) \tag{20}$$

which leads to:

$$J(s)G(s)=\frac{1}{E^2(s)} \tag{21}$$

where

$$E(s)=\frac{\mathbb{F}\left[\dfrac{\mathrm{d}^{\upsilon}\varepsilon(\tau)}{\mathrm{d}\tau^{\upsilon}}\right]}{\mathbb{F}\left[\varepsilon(t)\right]}=\frac{\mathbb{F}\left[\dfrac{\mathrm{d}^{\upsilon}\sigma(\tau)}{\mathrm{d}\tau^{\upsilon}}\right]}{\mathbb{F}\left[\sigma(t)\right]}$$

represents the parameter related to the corresponding general FD operators, *i. e.*, $E(s)=s^2$, $E(s)=s/(s-1)$, $E(s)=\mathrm{M}(\upsilon)s/[s+p(\upsilon)]$, and $E(s)=s/(s-1)$.

The creep and relaxation processes for the general fractional-order Maxwell models

A general fractional-order Maxwell model (GFMM) is that the branch of the spring and dashpot is in series [7].

The constitutive equations (CE) for the GFMM are given as:

$$\frac{1}{\gamma}\frac{\mathrm{d}^{\upsilon}\sigma(t)}{\mathrm{d}t^{\upsilon}}+\frac{\sigma(t)}{\Xi}=\frac{\mathrm{d}^{\upsilon}\varepsilon(t)}{\mathrm{d}t^{\upsilon}} \tag{22}$$

Substituting eqs. (1), (3), (5), (6), (8), (9), (11), and (12) into eq. (22) yields:

$$\begin{aligned}
&\frac{1}{\gamma}{}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t),\quad \frac{1}{\gamma}{}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t),\\
&\frac{1}{\gamma}{}^{\mathrm{LC}}_{e(-)}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{LC}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t),\quad \frac{1}{\gamma}{}^{\mathrm{RL}}_{e(-)}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{RL}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t),\\
&\frac{1}{\gamma}{}^{\mathrm{LC}}_{e(+)}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{LC}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t),\quad \frac{1}{\gamma}{}^{\mathrm{RL}}_{e(+)}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}^{\mathrm{RL}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t),
\end{aligned} \tag{23a,b,c,d,e,f}$$

$$\frac{1}{\gamma}\,{}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\varepsilon(t),\quad \frac{1}{\gamma}\,{}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}={}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\varepsilon(t) \tag{23g,h}$$

If the material is subjected to the one-step stress history $\sigma(t) = \phi(t)\sigma(0)$, where $\phi(t)$ is the unit step function [23], the creep differential equations (CDE) for the GFMM are written as.

With the use of eq. (12), we have:

$$\frac{\phi(t)\sigma(0)}{\Xi}={}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t),\quad \frac{\mathrm{M}(\upsilon)\sigma(0)}{\gamma}\mathrm{e}^{(-p(\upsilon)t)}+\frac{\phi(t)\sigma(0)}{\Xi}={}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t), \tag{24a,b}$$

$$\frac{\phi(t)\sigma(0)}{\Xi}={}_{e(-)}^{\mathrm{LC}}D_t^{(1)}\varepsilon(t),\quad \frac{\sigma(0)\mathrm{e}^{-t}}{\gamma}+\frac{\phi(t)\sigma(0)}{\Xi}={}_{e(-)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\varepsilon(t), \tag{24c,d}$$

$$\frac{\phi(t)\sigma(0)}{\Xi}={}_{e(+)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\varepsilon(t),\quad \frac{\sigma(0)\mathrm{e}^{t}}{\gamma}+\frac{\phi(t)\sigma(0)}{\Xi}={}_{e(+)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\varepsilon(t), \tag{24e,f}$$

$$\frac{\phi(t)\sigma(0)}{\Xi}={}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\varepsilon(t),\quad \frac{\sigma(0)}{\gamma t}+\frac{\phi(t)\sigma(0)}{\Xi}={}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\varepsilon(t) \tag{24g,h}$$

which deduce the corresponding CCF as:

$$\begin{gathered}
J(t)=\left[\frac{1}{\gamma}+\frac{1}{\mathrm{M}(\upsilon)\Xi}\right]+\frac{p(\upsilon)t}{\mathrm{M}(\upsilon)\Xi},\quad J(t)=\left[\frac{1}{\gamma}+\frac{1}{\Xi\mathrm{M}(\upsilon)}\right]+\frac{p(\upsilon)t}{\Xi\mathrm{M}(\upsilon)},\\
J(t)=\left(\frac{1}{\Xi}+\frac{1}{\gamma}\right)+\frac{t}{\Xi},\quad J(t)=\left(\frac{1}{\gamma}+\frac{1}{\Xi}\right)+\frac{t}{\Xi},\quad J(t)=\left(\frac{1}{\Xi}+\frac{1}{\gamma}\right)-\frac{t}{\Xi},\\
J(t)=\left(\frac{1}{\gamma}+\frac{1}{\Xi}\right)-\frac{t}{\Xi},\quad J(t)=\frac{t^2}{\Xi\Gamma(1+2)}+\frac{1}{\gamma},\quad J(t)=\frac{1}{\gamma}+\frac{t^2}{\Xi\Gamma(1+2)}
\end{gathered} \tag{25a,b,c,d,e,f,g,h}$$

If the material is subjected to the one-step strain history $\varepsilon(t) = \phi(t)\varepsilon(0)$, the relaxation differential equations (RDE) for the GFMM are given as:

$$\begin{gathered}
\frac{1}{\gamma}\,{}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\sigma(t)+\frac{\sigma(t)}{\Xi}=0,\quad \frac{1}{\gamma}\,{}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\sigma(t)+\frac{\sigma(t)}{\Xi}=\mathrm{M}(\upsilon)\varepsilon(0)\mathrm{e}^{[-p(\upsilon)t]},\\
\frac{1}{\gamma}\,{}_{e(-)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=0,\quad \frac{1}{\gamma}\,{}_{e(-)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=\varepsilon(0)\mathrm{e}^{-t},\\
\frac{1}{\gamma}\,{}_{e(+)}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=0,\quad \frac{1}{\gamma}\,{}_{e(+)}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=\varepsilon(0)\mathrm{e}^{t},\\
\frac{1}{\gamma}\,{}_{\oplus}^{\mathrm{LC}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=0,\quad \frac{1}{\gamma}\,{}_{\oplus}^{\mathrm{RL}}\mathrm{D}_t^{(1)}\sigma(t)+\frac{\sigma(t)}{\Xi}=\frac{\varepsilon(0)}{t}
\end{gathered} \tag{26a,b,c,d,e,f,g,h}$$

which leads to the corresponding RMF:

$$G(t)=\frac{\mathrm{M}(\upsilon)\Xi\gamma}{\mathrm{M}(\upsilon)\Xi+\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)}{\mathrm{M}(\upsilon)\Xi+\gamma}},\quad G(t)=\frac{\mathrm{M}(\upsilon)\Xi\gamma}{\mathrm{M}(\upsilon)\Xi+\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)}{\mathrm{M}(\upsilon)\Xi+\gamma}}, \tag{27a,b}$$

$$G(t)=\frac{\Xi\gamma}{\Xi+\gamma}\mathrm{e}^{-\frac{\gamma}{\Xi+\gamma}},\quad G(t)=\frac{\Xi\gamma}{\Xi+\gamma}\mathrm{e}^{-\frac{\gamma}{\Xi+\gamma}},$$
$$G(t)=\frac{\Xi\gamma}{\Xi+\gamma}\mathrm{e}^{\frac{\gamma}{\Xi+\gamma}},\quad G(t)=\frac{\Xi\gamma}{\Xi+\gamma}\mathrm{e}^{\frac{\gamma}{\Xi+\gamma}},\tag{27c,d,e,f,g,h}$$
$$G(t)=\gamma\cos\sqrt{\frac{\gamma}{\Xi}}t,\quad G(t)=\gamma\cos\sqrt{\frac{\gamma}{\Xi}}t$$

The creep and relaxation processes for the general fractional-order Kelvin-Voigt models

A general fractional-order Kelvin-Voigt model (GFKVM) is that the branch of the spring and dashpot is in parallel [7].

The CE for the GFMM are given as:

$$\Xi\frac{\mathrm{d}^{\upsilon}\varepsilon(t)}{\mathrm{d}t^{\upsilon}}+\gamma\varepsilon(t)=\sigma(t)\tag{28}$$

Thus, we reduce to the CE within the GFD as follows:

$$\Xi\,{}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),\quad \Xi\,{}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),$$
$$\Xi\,{}^{\mathrm{LC}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),\quad \Xi\,{}^{\mathrm{RL}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),\tag{29a,b,c,d,e,f,g,h}$$
$$\Xi\,{}^{\mathrm{LC}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),\quad \Xi\,{}^{\mathrm{RL}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),$$
$$\Xi\,{}^{\mathrm{LC}}_{\oplus}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t),\quad \Xi\,{}^{\mathrm{RL}}_{\oplus}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\sigma(t)$$

If the material is subjected to the one-step stress history $\sigma(t)=\phi(t)\sigma(0)$, the CDE for the GFKVM are presented as:

$$\Xi\,{}^{\mathrm{LC}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),\quad \Xi\,{}^{\mathrm{RL}}\mathrm{D}_t^{(\upsilon)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),$$
$$\Xi\,{}^{\mathrm{LC}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),\quad \Xi\,{}^{\mathrm{RL}}_{e(-)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),\tag{30a,b,c,d,e,f,g,h}$$
$$\Xi\,{}^{\mathrm{LC}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),\quad \Xi\,{}^{\mathrm{RL}}_{e(+)}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),$$
$$\Xi\,{}^{\mathrm{LC}}_{\oplus}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0),\quad \Xi\,{}^{\mathrm{RL}}_{\oplus}\mathrm{D}_t^{(1)}\varepsilon(t)+\gamma\varepsilon(t)=\phi(t)\sigma(0)$$

which yields the corresponding CCF as follows:

$$J(t)=\frac{1}{\mathrm{M}(\upsilon)+\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)t}{\mathrm{M}(\upsilon)+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)t}{\mathrm{M}(\upsilon)+\gamma}}+\frac{\mathrm{M}(\upsilon)}{\gamma\left[\mathrm{M}(\upsilon)+\gamma\right]}\mathrm{e}^{-\frac{\gamma p(\upsilon)t}{\mathrm{M}(\upsilon)+\gamma}},$$
$$J(t)=\frac{1}{\mathrm{M}(\upsilon)+\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)t}{\mathrm{M}(\upsilon)+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{-\frac{\gamma p(\upsilon)t}{\mathrm{M}(\upsilon)+\gamma}},\tag{31a,b,c}$$
$$J(t)=\frac{1}{1+\gamma}\mathrm{e}^{-\frac{\gamma t}{1+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{-\frac{\gamma t}{1+\gamma}}+\frac{1}{\gamma(1+\gamma)}\mathrm{e}^{-\frac{\gamma t}{1+\gamma}},$$

$$J(t)=\frac{1}{1+\gamma}\mathrm{e}^{-\frac{\gamma t}{1+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{-\frac{\gamma t}{1+\gamma}},$$

$$J(t)=\frac{1}{1+\gamma}\mathrm{e}^{\frac{\gamma t}{1+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{\frac{\gamma t}{1+\gamma}}+\frac{1}{\gamma(1+\gamma)}\mathrm{e}^{\frac{\gamma t}{1+\gamma}},$$

$$J(t)=\frac{1}{1+\gamma}\mathrm{e}^{\frac{\gamma t}{1+\gamma}}+\frac{t}{\gamma}-\frac{1}{\gamma}\mathrm{e}^{\frac{\gamma t}{1+\gamma}},\quad J(t)=\frac{1}{\gamma}\cos\sqrt{\gamma t}+\frac{1}{2\gamma^2}\left(\mathrm{e}^{\gamma t}-\frac{2}{t}+\mathrm{e}^{-\gamma t}\right), \tag{31d,e,f,g,h}$$

$$J(t)=\frac{1}{2\gamma^2}\left(\mathrm{e}^{\gamma t}-\frac{2}{t}+\mathrm{e}^{-\gamma t}\right)$$

If the material is subjected to the one-step strain history $\varepsilon(t)=\phi(t)\varepsilon(0)$, the RDE for the GFKVM are as follows:

$$\gamma\phi(t)\varepsilon(0)=\sigma(t),\quad \Xi\phi(t)\varepsilon(0)\mathrm{M}(\upsilon)\mathrm{e}^{[-p(\upsilon)t]}+\gamma\phi(t)\varepsilon(0)=\sigma(t),$$

$$\gamma\phi(t)\varepsilon(0)=\sigma(t),\quad \Xi\phi(t)\varepsilon(0)\mathrm{e}^{-t}+\gamma\phi(t)\varepsilon(0)=\sigma(t),$$

$$\gamma\phi(t)\varepsilon(0)=\sigma(t),\quad \Xi\phi(t)\varepsilon(0)\mathrm{e}^{t}+\gamma\phi(t)\varepsilon(0)=\sigma(t), \tag{32a,b,c,d,e,f,g,h}$$

$$\gamma\phi(t)\varepsilon(0)=\sigma(t),\quad \Xi\frac{\phi(t)\varepsilon(0)}{t}+\gamma\phi(t)\varepsilon(0)=\sigma(t)$$

which results in the corresponding RMF as follows:

$$G(t)=\gamma,\quad G(t)=\Xi\mathrm{M}(\upsilon)\mathrm{e}^{[-p(\upsilon)t]}+\gamma,\quad G(t)=\gamma,\quad G(t)=\Xi\mathrm{e}^{-t}+\gamma,$$

$$G(t)=\gamma,\quad G(t)=\Xi\mathrm{e}^{t}+\gamma,\quad G(t)=\gamma,\quad G(t)=\frac{\Xi}{t}+\gamma \tag{33a,b,c,d,e,f,g,h}$$

Conclusions and remarks

From mathematics and physics view point, we considered the general Riemann-Liouville and Caputo-Liouville FD without singular kernel to characterize the rheological properties of the real materials. The fractional-order Maxwell and Kelvin-Voigt models were discussed with the aid of the Laplace transform. The proposed approaches are efficient for analyzing and modelling the creep and relaxation behaviors in the linear viscoelastic materials.

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Nomenclature

t – time, [s]

Greek symbols

$\varepsilon(t)$ – strain, [–]
υ – fractional dimension, [–]
$\sigma(t)$ – stress, [Pa]

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